PART B

Unit - VII

Probability Theory - 2

7.1 Random variables - Introduction

In a random experiment, if a real variable is associated with every outcome then it is called a *random variable* or *stochastic variable.* In other words a random variable is a function that assigns a real number to every sample point in the sample space of a random experiment. Random variables are usually denoted by $X, Y, Z \cdots$ and it may be noted that different random variables may be associated with the same sample space S. The set of all real numbers of a random variable X is called the *range* of X.

Example - 1 While tossing a coin, suppose that the value 1 is associated for the outcome 'head' and 0 for the outcome 'tail'. We have the sample space

$S = \{H, T\}$ and if X is the random variable then $X(H) = 1$ & $X(T) = 0$

Range of $X = \{0, 1\}$

Example - 2 Suppose a coin is tossed twice, we shall associate two different random variables X, Y as follows where we have the sample space

$S = \{HH, HT, TH, TT\}$

X = Number of 'heads' in the outcome.

The association of the elements in S to X is as follows.

Out come	HH	HT	TH	TT
Random variable X	2	1	1	0

Range of $X = \{0, 1, 2\}$

Suppose Y = Number of 'tails' in the outcome.

Out come	HH	HT	TH	TT
Random variable X	0	1	1	2

Range of $Y = \{0, 1, 2\}$

Example - 3 Let the random experiment be throwing a pair of 'die' and the sample space S associated with it is the set of all pairs of numbers chosen from 1 to 6.

That is, $S = \left\{ (x, y) / x, y \text{ belong the set of nos. } 1, 2, 3, 4, 5, 6 \right\}$

To each outcome (x, y) of S let us associate a random variable $X = x + y$

The range of $X = \{2, 3, 4, \cdots 12\}$ corresponding to the 36 outcomes in S namely $(1, 1), (1, 2) \cdots (1, 6), (2, 1) \cdots (6, 6)$

We have $X(1, 1) = 2$; $X(1, 2) = 3 = X(2, 1)$;

$X(1, 3) = 4 = X(3, 1) = X(2, 2)$ etc.

7.2 Discrete and Continuous random variables - Definitions

If a random variable takes finite or countably infinite number of values then it is called a ***discrete random variable***. Here countably infinite means a sequence of real numbers. It is evident that a discrete random variable will have finite or countably infinite range.

If a random variable takes non countable infinite number of values then it is called a ***non discrete*** or ***continuous random variable***. Equivalently we can say that, if the range of a random variable X is an interval of real numbers then X is a continuous random variable. A continuous random variable can assume any value in the interval of real numbers.

Example - 1

(a) Tossing a coin and observing the outcome.

(b) Tossing coins and observing the number of heads turning up.

(c) Throwing a'die' and observing the numbers on the face.

These are some of the examples of a discrete random variable.

Example - 2

(a) Weight of articles.

(b) Length of nails produced by a machine.

(c) Observing the pointer on a speedometer / voltmeter.

(d) Conducting a survey on the life of electric bulbs.

These are some of the examples of a continuous random variable.

Generally counting problems correspond to discrete random variables and measuring problems lead to continuous random variables.

According to the category of the random variable we have two types of probability distributions.

7.3 Discrete probability distribution - Definitions

If for each value x_i of a discrete random variable X, we assign a real number $p(x_i)$ such that

(1) $p(x_i) \geq 0$ (2) $\sum_i p(x_i) = 1$

then the function $p(x)$ is called a ***probability function.***
If the probability that X take the values x_i is p_i, then

$$P(X = x_i) = p_i \text{ or } p(x_i).$$

The set of values $[x_i, p(x_i)]$ is called a ***discrete (finite) probability distribution*** of the discrete random variable X. The function $P(X)$ is called the ***probability density function*** $(p \cdot d \cdot f)$ or the ***probability mass function*** $(p \cdot m \cdot f)$

The distribution function $f(x)$ defined by

$f(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$, x being an integer is called the

cumulative distribution function $(c \cdot d \cdot f)$

We are already familiar with the definitions of mean $(\bar{x})$ and variance (σ^2) for a general frequency distribution. That is

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} \quad \text{and} \quad \sigma^2 = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}$$

We can as well define the mean and variance of the discrete probability distribution. In this case $p(x_i)$ corresponds to f_i and we have $\sum f_i = \sum p(x_i) = 1$.

The mean and variance of the discrete probability distribution is defined as follows.

Mean $(\mu) = \sum_i x_i \cdot p(x_i)$ ...(1)

Variance $(V) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$ ...(2)

Standard deviation $(\sigma) = \sqrt{V}$ ...(3)

Note : *Variance can also be put in the form*

$$V = \sum_i x_i^2 \, p(x_i) - [\sum_i x_i p(x_i)]^2$$

WORKED PROBLEMS

1. *A coin is tossed twice. A random variable X represent the number of heads turning up. Find the discrete probability distribution for X. Also find its mean and variance.*

>> $S = \{HH,\ HT,\ TH,\ TT\}$. The association of the elements of S to the random variable X are respectively 2, 1, 1, 0.

Now, $P(HH) = \frac{1}{4},\ P(HT) = \frac{1}{4},\ P(TH) = \frac{1}{4},\ P(TT) = \frac{1}{4}$

$P\ (X = 0, \text{ i.e., no head}) = P(TT) = \frac{1}{4}$

$P\ (X = 1, \text{ i.e., 1 head}) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{2}$

$P\ (X = 2, \text{ i.e., 2 heads}) = P(HH) = \frac{1}{4}$

The discrete probability distribution for X is as follows.

$X = x_i$	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

We observe that $p(x_i) > 0$ and $\sum p(x_i) = 1$

Mean $= \mu = \sum x_i \cdot p(x_i) = 0 + \frac{1}{2} + \frac{1}{2} = 1$

Variance $= V = \sum (x_i - \mu)^2 \cdot p(x_i)$

$$V = (0-1)^2 \cdot \frac{1}{4} + (1-1)^2 \cdot \frac{1}{2} + (2-1)^2 \cdot \frac{1}{4} = \frac{1}{2}$$

Thus we have **Mean = 1** and **Variance = 1/2**

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2. *A random experiment of tossing a 'die' twice is performed. Random variable X on this sample space is defined to be the sum of the two numbers turning up on the toss. Find the discrete probability distribution for the random variable X and compute the corresponding mean and standard deviation.*

>> $S = \{(x, y) \text{ where } x = 1, 2, \ldots 6\ ;\ y = 1, 2, \ldots 6\}$

i.e., $S = \{(1,1)(1,2)(1,3), \ldots (6,4), (6,5), (6,6)\}$

Number of elements in $S = n(S) = 36$.

The set of values of the random variable X defined as the sum of two numbers on the face of the 'die' are $\{2,\ 3,\ 4,\ \ldots 10,\ 11,\ 12\}$

The association of the elements of S to the sample variable X are tabulated.

Elements of S or Events E	$X = x_i$	Total No. of events : $n(E)$
1. (1,1)	$x_1 = 2$	1
2. (1,2)(2,1)	$x_2 = 3$	2
3. (1,3)(3,1)(2,2)	$x_3 = 4$	3
4. (1,4)(4,1)(2,3)(3,2)	$x_4 = 5$	4
5. (1,5)(5,1)(2,4)(4,2)(3,3)	$x_5 = 6$	5
6. (1,6)(6,1)(2,5)(5,2)(4,3)(3,4)	$x_6 = 7$	6
7. (2,6)(6,2)(3,5)(5,3)(4,4)	$x_7 = 8$	5
8. (3,6)(6,3)(4,5)(5,4)	$x_8 = 9$	4
9. (4,6)(6,4)(5,5)	$x_9 = 10$	3
10. (5,6)(6,5)	$x_{10} = 11$	2
11. (6,6)	$x_{11} = 12$	1
Total		36

Now $p(x_1) = \frac{n(E)}{n(S)} = \frac{1}{36}$; $p(x_2) = \frac{2}{36}$ etc.

The discrete probability distribution for X is as follows.

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$p(x_i) > 0 \text{ and } \sum_i p(x_i) = 1.$$

Mean $= \mu = \sum x_i \cdot p(x_i) = \frac{252}{36} = 7.$

Variance $= V = \sum_i (x_i - \mu)^2 \cdot p(x_i) = \frac{210}{36} = \frac{35}{6}$

Thus Mean = 7 and S.D = $\sqrt{V} = \sqrt{35/6}$

3. *Show that the following distribution represents a discrete probability distribution. Find the mean and variance.*

x	10	20	30	40
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

>> We observe that $p(x) > 0$ for all x and $\sum p(x) = 1$

Mean $= \mu = \sum p(x) \cdot x = \dfrac{10+60+90+40}{8} = 25$

Variance $= V = \sum (x-\mu)^2 \cdot p(x)$

$$= 225 \cdot \frac{1}{8} + 25 \cdot \frac{3}{8} + 25 \cdot \frac{3}{8} + 225 \cdot \frac{1}{8}$$

$$V = \frac{600}{8} = 75$$

Thus **Mean = 25 and Variance = 75**

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4. *Find the value of k such that the following distribution represents a finite probability distribution. Hence find its mean and standard deviation. Also find* $p(x \le 1)$, $p(x > 1)$ *and* $p(-1 < x \le 2)$

x	-3	-2	-1	0	1	2	3
$p(x)$	k	$2k$	$3k$	$4k$	$3k$	$2k$	k

>> We must have $p(x) \ge 0$ for all x and $\sum p(x) = 1$. The first condition is satisfied if $k \ge 0$ and the second condition requires that,

$$k + 2k + 3k + 4k + 3k + 2k + k = 1 \text{ or } 16k = 1 \quad \therefore \quad k = 1/16$$

The discrete / finite probability distribution is as follows.

x	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Mean $= \mu = \sum x \cdot p(x) = \dfrac{1}{16}(-3-4-3+0+3+4+3) = 0$

Variance $= V = \sum (x-\mu)^2 \cdot p(x)$

$$= \frac{1}{16}(9+8+3+0+3+8+9) = \frac{40}{16} = \frac{5}{2}$$

Thus k **= 1/16, Mean = 0 and S.D =** $\sqrt{5/2}$

Also, $p(x \le 1) = p(-3)+p(-2)+p(-1)+p(0)+p(1) =$ **13/16**

$p(x > 1) = p(2)+p(3) =$ **3/16**

$p(-1 < x \le 2) = p(0)+p(1)+p(2) =$ **9/16**

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5. *The p.d.f of a variate X is given by the following table.*

x	0	1	2	3	4	5	6
$P(x)$	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

For what value of K, this represents a valid probability distribution ?

Also find $P(x \geq 5)$ *and* $P(3 < x \leq 6)$

>> The probability distribution is valid if $P(x) \geq 0$ and $\sum P(x) = 1$

Hence we must have $K \geq 0$ and $K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

i.e., $49K = 1$ or $K = 1/49$

Also $P(x \geq 5) = P(5) + P(6) = 11K + 13K = 24K =$ **24/49**

$P(3 < x \leq 6) = P(4) + P(5) + P(6) = 33K =$ **33/49**

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6. *The probability distribution of a finite random variable X is given by the following table.*

x_i	-2	-1	0	1	2	3
$P(x_i)$	0.1	K	0.2	$2K$	0.3	K

Find the value of K, mean and variance.

>> We must have $P(x_i) \geq 0$ and $\sum P(x_i) = 1$ for a probability distribution.

$\sum P(x_i) = 1$ requires $4K + 0.6 = 1$ $\therefore$ $\mathbf{K = 0.1}$

Mean $(\mu) = \sum x_i P(x_i) = -0.2 - 0.1 + 0.2 + 0.6 + 0.3 = 0.8$

Variance $(\sigma^2) = \sum x_i^2 P(x_i) - \mu^2$

$$\sigma^2 = (0.4 + 0.1 + 0.2 + 1.2 + 0.9) - (0.8)^2 = 2.16$$

Thus Mean = 0.8 and Variance = 2.16

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7. *A random variable X has the following probability function for various values of x*

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k (ii) Evaluate $P(x < 6)$, $P(x \geq 6)$ *and* $P(3 < x \leq 6)$

Also find the probability distribution and the distribution function of X

>> We must have $P(x) \geq 0$ and $\sum P(x) = 1$

The first condition is satisfied for $k \geq 0$ and we have to find k such that $\sum P(x) = 1$

i.e., $0 + k + 2k + 2k - 3k + k^2 + 2k^2 + (7k^2 + k) = 1$

i.e., $10k^2 + 9k - 1 = 0$ or $(10k-1)(k+1) = 0$ or $k = 1/10$ and $k = -1$

If $k = -1$ the first condition fails and hence $k \neq -1$ $\therefore$ $\mathbf{k = 1/10}$

Hence we have the following table.

x	0	1	2	3	4	5	6	7
$P(x)$	0	1/10	1/5	1/5	3/10	1/100	1/50	17/100

Now $P(x < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

i.e., $P(x < 6) = 0 + 1/10 + 1/5 + 1/5 + 3/10 + 1/100 = 81/100 = \mathbf{0.81}$

$P(x \geq 6) = P(6) + P(7)$

i.e., $P(x \geq 6) = 1/50 + 17/100 = 19/100 = \mathbf{0.19}$

$P(3 < x \leq 6) = P(4) + P(5) + P(6)$

i.e., $P(3 < x \leq 6) = 3/10 + 1/100 + 1/50 = 33/100 = \mathbf{0.33}$

The Probability distribution is as follows.

x	0	1	2	3	4	5	6	7
$P(x)$	**0**	**0.1**	**0.2**	**0.2**	**0.3**	**0.01**	**0.02**	**0.17**

The distribution function of X is $f(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$ is also called cumulative distribution function and the same is as follows.

x	0	1	2	3	4	5	6	7
$f(x)$	0	0 + 0.1 = **0.1**	0.1 + 0.2 = **0.3**	0.3 + 0.2 = **0.5**	0.5 + 0.3 = **0.8**	0.8 + 0.01 = **0.81**	0.81 + 0.02 = **0.83**	0.83 + 0.17 = **1**

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8. *A random variable X take the values* $-3, -2, -1, 0, 1, 2, 3$ *such that* $P(X = 0) = P(X < 0)$ *and* $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$. *Find the probability distribution.*

>> Let the distribution $[X, P(X)]$ be as follows.

X	−3	−2	−1	0	1	2	3
$P(X)$	p_1	p_2	p_3	p_4	p_5	p_6	p_7

By data, $P(X=0) = P(X<0)$

$\Rightarrow \quad P(X=0) = P(X=-1)+P(X=-2)+P(X=-3)$

i.e., $\quad p_4 = p_3+p_2+p_1$ ...(1)

Also we have by data,

$$p_1 = p_2 = p_3 = p_5 = p_6 = p_7 \qquad ...(2)$$

Further we must have,

$$p_1+p_2+p_3+p_4+p_5+p_6+p_7 = 1 \qquad ...(3)$$

Using (2) in (1) we get $3p_1 = p_4$

Using (2) in (3) we get $6p_1+p_4 = 1$. But $p_4 = 3p_1$

$\therefore \quad 9p_1 = 1$ or $p_1 = 1/9$. Hence $p_4 = 3\cdot(1/9) = 1/3$

Thus the probability distribution is as follows.

X	-3	-2	-1	0	1	2	3
$P(X)$	**1/9**	**1/9**	**1/9**	**1/3**	**1/9**	**1/9**	**1/9**

9. *From a sealed box containing a dozen apples it was found that 3 apples are perished. Obtain the probability distribution of the number of perished apples when 2 apples are drawn at random. Also find the mean and variance of this distribution.*

>> Let X be the number of perished apples. Since 2 were drawn we have $X = 0, 1, 2$. 2 out of 12 can be selected in 12_{C_2} ways. 9 are good apples and 3 are perished apples. Hence we have

$$P(X=0) = \text{Probability of getting 0 perished apple} = \frac{3_{C_0}\cdot 9_{C_2}}{12_{C_2}} = \frac{6}{11}$$

$$P(X=1) = \text{Probability of getting 1 perished apple} = \frac{3_{C_1}\cdot 9_{C_1}}{12_{C_2}} = \frac{9}{22}$$

$$P(X=2) = \text{Probability of getting 2 perished apples} = \frac{3_{C_2}\cdot 9_{C_0}}{12_{C_2}} = \frac{1}{22}$$

The probability distribution is as follows.

$X = x_i$	0	1	2
$P(X) = p_i$	**6/11**	**9/22**	**1/22**

Mean $(\mu) = \sum x_i p_i = 0 + \frac{9}{22} + \frac{2}{22} = \frac{11}{22} = \frac{1}{2}$

Variance $(V) = \sum x_i^2 p_i - \mu^2$

$$= \left(0 + \frac{9}{22} + \frac{4}{22}\right) - \frac{1}{4} = \frac{13}{22} - \frac{1}{4} = \frac{15}{44}$$

Thus **Mean = 1/2 and Variance = 15/44**

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10. *If the random variable X take the values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, find the probability distribution function of X.*

>> Let the distribution $[X, P(X)]$ be as follows.

X	1	2	3	4
$P(X)$	p_1	p_2	p_3	p_4

By data, $2p_1 = 3p_2 = p_3 = 5p_4$ ...(1)

We also have, $p_1 + p_2 + p_3 + p_4 = 1$ ...(2)

From (1), $p_2 = \frac{2}{3}p_1,\ p_3 = 2p_1,\ p_4 = \frac{2}{5}p_1$

Hence (2) becomes, $p_1 + \frac{2}{3}p_1 + 2p_1 + \frac{2}{5}p_1 = 1$ or $\frac{61}{15}p_1 = 1 \therefore p_1 = \frac{15}{61}$

Hence we get $p_2 = \frac{10}{61},\ p_3 = \frac{30}{61},\ p_4 = \frac{6}{61}$

The probability distribution $P(X)$ and the cumulative distribution $f(x) = \sum_{i=1}^{x} p(x_i)$ is as follows.

$X = x_i$	1	2	3	4
$P(X)$	**15/61**	**10/61**	**30/61**	**6/61**
$f(x)$	**15/61**	**25/61**	**55/61**	**61/61 = 1**

11. *A random variable X has $p(x) = 2^{-x}$, $x = 1, 2, 3 \cdots$ Show that $p(x)$ is a probability function. Also find $p(X \text{ even})$, $p(X \text{ being divisible by } 3)$ and $p(X \geq 5)$*

>> $p(x) = 2^{-x} = \dfrac{1}{2^x}$ Evidently $p(x) > 0$ for all x.

$$\sum_i p(x_i) = \sum_{x=1}^{\infty} \frac{1}{2^x} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

RHS is a geometric series of the form $a + ar + ar^2 + \cdots$ whose sum to infinity is $a/1 - r$ where we have $a = 1/2 = r$

$$\therefore \quad \sum_i p(x_i) = \frac{1/2}{1-(1/2)} = \frac{1/2}{1/2} = 1$$

Hence $p(x) = 2^{-x}$ **is a probability function.**

Case - (i) : $p(X \text{ even}) = \displaystyle\sum_{x=2,4,6\cdots}^{\infty} p(x) = \sum_{2,4,6\cdots}^{\infty} \frac{1}{2^x}$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots.$$

$$= \frac{1/2^2}{1-(1/2^2)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Thus $p(X \text{ even}) = 1/3$

Case - (ii) : $p(X \text{ being divisible by } 3) = \displaystyle\sum_{x=3,6,9\cdots}^{\infty} \frac{1}{2^x}$

ie., $= \dfrac{1}{2^3} + \dfrac{1}{2^6} + \dfrac{1}{2^9} + \cdots$

$$= \frac{1/2^3}{1-(1/2^3)} = \frac{1/8}{7/8} = \frac{1}{7}$$

Thus $p(X \text{ divisible by } 3) = 1/7$

Case - (iii) : $p(X \geq 5) = 1 - p(X < 5)$

$$p(X \geq 5) = 1 - \sum_{i=1}^{4} p(x_i) = 1 - \sum_{x=1}^{4} \frac{1}{2^x}$$

$$p\,(X \geq 5) = 1 - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right)$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) = 1 - \frac{15}{16} = \frac{1}{16}$$

Thus $p(X \geq 5) = 1/16$

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12. *If X is a discrete random variable taking values* $1, 2, 3, \cdots$ *with* $p(x) = 1/2 \cdot (2/3)^x$ *, find* $p\,($*X being an odd number*$)$ *by first establishing that* $p\,(x)$ *is a probability function.*

>> $$\sum_i p(x_i) = \sum_{x=1}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^x$$

$$= \frac{1}{2}\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right]$$

$$= \frac{1}{2} \cdot \frac{2/3}{1-(2/3)} = \frac{1}{2} \cdot \frac{2/3}{1/3} = 1$$

$\therefore$ **$p(x)$ is a probability function.**

Next $p\,(X$ being an odd number$) = \sum_{x=1,3,5\cdots}^{\infty} p(x) = \sum_{x=1,3,5\cdots}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^x$

$$= \frac{1}{2}\left[\frac{2}{3} + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^5 + \cdots\right].$$

$$= \frac{1}{2}[a + ar + ar^2 + \cdots] \bullet \text{where } a = 2/3,\ r = (2/3)^2$$

$$= \frac{1}{2} \cdot \frac{a}{1-r} = \frac{1}{2} \cdot \frac{2/3}{1-(2/3)^2} = \frac{1}{2} \cdot \frac{2/3}{5/9} = \frac{3}{5}$$

Thus **$p(X$ being an odd number$) = 3/5$**

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13. *The range of a random variable* $X = \{1, 2, 3 \cdots\}$ *& the probabilities of* X *are such that*

$$P(X = k) = \frac{\lambda^k}{k!} \text{ where } k = 1, 2, 3, \cdots.$$

Find the value of λ *and* $P(0 < X < 3)$

>> We must have $\sum P(X) = 1$

That is, $\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = 1$

or $\lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = 1$

i.e., $(e^{\lambda} - 1) = 1$ or $e^{\lambda} = 2 \Rightarrow \lambda = \log_e 2$

Also $P(0 < X < 3) = P(X = 1) + P(X = 2)$

$$= \frac{\lambda}{1!} + \frac{\lambda^2}{2!} \text{ where } \lambda = \log_e 2$$

Thus $P(0 < X < 3) = \log_e 2 + \frac{1}{2}(\log_e 2)^2$

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14. *X is a discrete random variable having $p(x)$ defined as follows.*

$$p(x) = \begin{cases} x/15 & \text{if } 1 \le x \le 5 \\ 0 & \text{if } x > 5 \end{cases}$$ *Show that $p(x)$ is a probability function.*

Find (i) $P(X = 1 \text{ or } 2)$ *(ii)* $P(1/2 < X < 5/2 \,/\, X > 1)$

>> The probability distribution is as follows.

x	1	2	3	4	5	6, 7, ...
$p(x)$	1/15	2/15	3/15	4/15	5/15	0

We have $p(x) \ge 0$ and $\sum_i p(x_i) = 1$

$\therefore$ $p(x)$ is a probability function.

Now $P(X = 1 \text{ or } 2) = P(X = 1) + P(X = 2)$

$$= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$$

Thus $P(X = 1 \text{ or } 2) = 1/5$

$$P(1/2 < X < 5/2 \,/\, X > 1) = \frac{P(1/2 < X < 5/2) \cap P(X > 1)}{P(X > 1)}$$

i.e.,

$$= \frac{P(X = 1 \text{ or } 2) \cap P(X > 1)}{1 - P(X \le 1)}$$

$$= \frac{P(X = 2)}{1 - P(X = 1)} = \frac{2/15}{1 - (1/15)} = \frac{2/15}{14/15} = \frac{1}{7}$$

Thus $P(1/2 < X < 5/2 \,/\, X > 1) = 1/7$

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15. *The range of a random variable* $X = \{1, 2, 3 \cdots n\}$ *and the probabilities of* X *are* kx. *Find the value of* k *and also compute the mean and variance of the probability distribution.*

>> By data the distribution $[X, P(X)]$ is as follows.

$X = x_i$	1	2	3	....	n
$P(X) = p_i$	k	$2k$	$3k$	....	nk

We must have $P(X) \geq 0$ and $\sum P(X) = 1$

This requires $k \geq 0$ and $k + 2k + 3k + \cdots + nk = 1$

i.e., $k(1+2+3+\cdots+n) = 1$

or $k \cdot \dfrac{n(n+1)}{2} = 1 \qquad \therefore \quad k = \dfrac{2}{n(n+1)}$

Mean $(\mu) = \sum x_i p_i$

$$= k + 2^2 k + 3^2 k + \cdots + n^2 k$$

$$= k(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$= \frac{2}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6} = \mathbf{\frac{2n+1}{3}}$$

Variance $(V) = \sum x_i^2 p_i - \mu^2$

$$= (1^2 \cdot k + 2^2 \cdot 2k + 3^2 \cdot 3k + \cdots + n^2 \cdot nk) - \mu^2$$

$$= k(1^3 + 2^3 + 3^3 + \cdots + n^3) - \mu^2$$

$$= k \cdot \frac{n^2(n+1)^2}{4} - \left(\frac{2n+1}{3}\right)^2$$

$$= \frac{2}{n(n+1)} \cdot \frac{n^2(n+1)^2}{4} - \frac{(2n+1)^2}{9}$$

$$= \frac{n(n+1)}{2} - \frac{(2n+1)^2}{9}$$

$$= \frac{1}{18}\left[9n(n+1) - 2(4n^2 + 4n + 1)\right] = \mathbf{\frac{n^2+n-2}{18}}$$

Thus **Mean** $= \mathbf{\dfrac{2n+1}{3}}$ **and Variance** $= \mathbf{\dfrac{n^2+n-2}{18}}$

7.4 Repeated Trials

A random experiment with only two possible outcomes categorized as *success* and *failure* is called a *Bernoulli trial* where the probability of success p is same for each trial.

7.41 Bernoulli's Theorem

The probability of x successes in n trials is equal to $n_{C_x} p^x q^{n-x}$ where p is the probability of success and q is the probability of failure.

Proof : Since p is the probability of success, the probability of x successes is $p \cdot p \cdot p \cdots x$ times $= p^x$

Also x successes imply $(n-x)$ failures and since q is the probability of failure, the probability of $(n-x)$ failures is q^{n-x}

By the multiplication rule the probability of the simultaneous happening is $p^x q^{n-x}$

But x successes in n trials can occur in n_{C_x} ways and all these cases are favourable to the event.

Hence by the addition rule, the probability of x successes out of n trials is given by

$$p^x q^{n-x} + p^x q^{n-x} + \cdots n_{C_x} \text{ times } = n_{C_x} p^x q^{n-x}$$

This proves Bernoulli's theorem.

An illustration of the theorem

Suppose a coin is tossed 3 times and we wish to know the probability of getting 2 heads.

The possibilities are as follows.

$$HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT$$

We see that there are eight possibilities out of which only 3 possibilities are favourable to the event. They are $HHT,\ HTH,\ THH$.

Thus the probability of getting 2 heads is 3/8.

Now we shall obtain the probability of the event by Bernoulli's theorem.

Number of trials $n = 3$

Number of successes (getting 2 heads) $x = 2$

Probability of success (p) = Probability of getting a head $= 1/2$

Probability of failure $(q) = 1 - p = 1/2$

$$\therefore \quad {}^{n}C_{x}\, p^{x} q^{n-x} = {}^{3}C_{2} \left(\frac{1}{2}\right)^{2} \left(\frac{1}{2}\right)^{1} = 3 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

Remark : *Bernoulli's theorem leads to* ***Binomial Distribution*** *and a limiting case of this distribution leads to* ***Poisson Distribution****.* ***These are Discrete Probability Distributions.***

We proceed to discuss these in detail.

7.5 Binomial Distribution

If p is the probability of success and q is the probability of failure, the probability of x successes out of n trials is given by

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

We form the following probability distribution of $[x, P(x)]$ where $x = 0, 1, 2, \cdots n$

x	0	1	2	$\cdots$	n
$P(x)$	q^{n}	${}^{n}C_{1}\, q^{n-1} p$	${}^{n}C_{2}\, q^{n-2} p^{2}$	$\cdots$	p^{n}

It may be observed that the value of $P(x)$ for different values $x = 0, 1, 2, \cdots n$ are the successive terms in the binomial expansion of $(q + p)^{n}$ and accordingly this distribution is called the ***Binomial Distribution*** or ***Bernoulli Distribution.***

$$\sum P(x) = q^{n} + {}^{n}C_{1}\, q^{n-1} p + {}^{n}C_{2}\, q^{n-2} p^{2} + \cdots + p^{n} = (q + p)^{n} = 1^{n} = 1$$

Hence $P(x)$ is a probability function.

7.51 Mean and Standard Deviation of the Binomial Distribution

$$\text{Mean } (\mu) = \sum_{x=0}^{n} x\, P(x)$$

$$\mu = \sum_{x=0}^{n} x \cdot {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n \cdot (n-1)!}{(x-1)!(n-x)!} p \cdot p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} (n-1)_{C_{(x-1)}} p^{x-1} q^{(n-1)-(x-1)}$$

$$\mu = np(q+p)^{n-1} = np$$

Mean $(\mu) = np$

$$\text{Variance } (V) = \sum_{x=0}^{n} x^2 P(x) - \mu^2 \qquad \ldots (1)$$

$$\text{Now } \sum_{x=0}^{n} x^2 P(x) = \sum_{x=0}^{n} [x(x-1)+x] P(x)$$

$$= \sum_{x=0}^{n} x(x-1) P(x) + \sum_{x=0}^{n} x P(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, n_{C_x} p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} \frac{n(n-1)\cdot(n-2)!}{(x-2)!(n-x)!} p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)![(n-2)-(x-2)]!} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} (n-2)_{C_{(x-2)}} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$\therefore \quad \sum x^2 P(x) = n(n-1)p^2 + np$$

Using this result in (1) along with $\mu = np$ we have

$$\text{Variance } (V) = \{n(n-1)p^2 + np\} - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

Hence variance $(V) = npq$

$$S.D\ (\sigma) = \sqrt{V} = \sqrt{npq}$$

Thus we have for the Binomial Distribution,

Mean $(\mu) = np$ and $S.D\ (\sigma) = \sqrt{npq}$

Note : *The usual notation* $\bar{x}$ *for the mean can also be used.*

7.6 Poisson Distribution

Poisson distribution is regarded as the limiting form of the binomial distribution when n is very large $(n \to \infty)$ and p the probability of success is very small $(p \to 0)$ so that np tends to a fixed finite constant say m.

We have in the case of binomial distribution, the probability of x successes out of n trials,

$$P(x) = {}^nC_x\, p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-\overline{x-1})}{x!} p^x q^{n-x}$$

$$= \frac{n \cdot n\left(1-\frac{1}{n}\right) n\left(1-\frac{2}{n}\right)\cdots n\left(1-\frac{x-1}{n}\right)}{x!} p^x q^{n-x}$$

$$= \frac{n^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots n\left(1-\frac{x-1}{n}\right)}{x!} p^x q^{n-x}$$

$$= \frac{(np)^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) q^n}{x!\,q^x}$$

But $\quad np = m\ ;\ q^n = (1-p)^n = \left(1-\frac{m}{n}\right)^n = \left\{\left(1-\frac{m}{n}\right)^{-n/m}\right\}^{-m}$

Denoting $-\frac{m}{n} = k$ we have,

$$q^n = \left\{(1+k)^{1/k}\right\}^{-m} \to e^{-m} \text{ as } n \to \infty \text{ or } k \to 0.$$

[**Note** : $\lim_{k \to 0} (1+k)^{1/k} = e$]

Further $q^x = (1-p)^x \to 1$ for a fixed x as $p \to 0$.

Also the factors $\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$ will all tend to 1 as $n \to \infty$

Thus we get $P(x) = \dfrac{m^x e^{-m}}{x!}$

This is known as the Poisson distribution of the random variable.

$P(x)$ is also called ***Poisson probability function*** and x is called a ***Poisson variate.***

The distribution of probabilities for $x = 0, 1, 2, 3, \cdots$ is as follows.

x	0	1	2	3	...
$P(x)$	e^{-m}	$\frac{m e^{-m}}{1!}$	$\frac{m^2 e^{-m}}{2!}$	$\frac{m^3 e^{-m}}{3!}$	

We have $P(x) \geq 0$ and

$$\sum_{x=0}^{\infty} P(x) = e^{-m} + \frac{m e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} + \cdots$$

$$= e^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right\}$$

$$= e^{-m} \cdot e^{m} = e^0 = 1. \qquad \therefore \quad \sum P(x) = 1$$

Hence $P(x)$ is a probability function.

7.61 Mean and Standard Deviation of the Poisson Distribution

$$\text{Mean } (\mu) = \sum_{x=0}^{\infty} x \cdot P(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{m^x e^{-m}}{x!} = \sum_{x=1}^{\infty} \frac{m^x e^{-m}}{(x-1)!}$$

$$\mu = m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= m e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right]$$

$$= m e^{-m} \cdot e^{m} = m \cdot e^0 = m$$

$\therefore$ **Mean** $(\mu) = m$

$$\text{Variance } (V) = \sum_{x=0}^{\infty} x^2 P(x) - \mu^2 \qquad \ldots(1)$$

$$\text{Now} \quad \sum x^2 P(x) = \sum [x(x-1) + x] \frac{m^x e^{-m}}{x!}$$

$$= \sum \frac{m^x e^{-m}}{(x-2)!} + \sum \frac{m^x e^{-m}}{(x-1)!}$$

$$\sum x^2 P(x) = m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$= m^2 e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots \right] + m$$

$$= m^2 e^{-m} \cdot e^m + m$$

i.e., $\sum x^2 P(x) = m^2 + m$

Using this result in (1) along with $\mu = m$ we have,

Variance $(V) = m^2 + m - (m)^2 = m$

$\therefore$ S.D $(\sigma) = \sqrt{V} = \sqrt{m}$

Thus we have for the Poisson distribution

Mean $(\mu) = m$ and S.D $(\sigma) = \sqrt{m}$

Further we can say that the ***mean and variance are equal for the Poission distribution.***

WORKED PROBLEMS

16. *Find the binomial probability distribution which has mean 2 and variance 4/3.*

>> We know that for the binomial distribution mean $= np$ and variance $= npq$

Hence $np = 2$ and $npq = 4/3$ by using the data.

Further $2q = 4/3$ or $q = 2/3$ $\quad\therefore\quad p = 1 - q = 1/3$

Since $np = 2$, we have $n(1/3) = 2$ $\quad\therefore\quad n = 6$

The binomial probability function $P(x) = n_{C_x}\, p^x q^{n-x}$ becomes

$$P(x) = 6_{C_x} (1/3)^x (2/3)^{6-x}$$

The distribution of probabilities is as follows.

x	0	1	2	3
$P(x)$	$(2/3)^6$	$6_{C_1}(1/3)(2/3)^5$	$6_{C_2}(1/3)^2(2/3)^4$	$6_{C_3}(1/3)^3(2/3)^3$
		4	5	6
		$6_{C_4}(1/3)^4(2/3)^2$	$6_{C_5}(1/3)^5(2/3)$	$(1/3)^6$

17 *When a coin is tossed 4 times, find the probability of getting (i) exactly one head (ii) atmost 3 heads (iii) atleast 2 heads*

>> $p = P(H) = 1/2 = 0.5,\ q = 0.5\ ;\ n = 4$

We know that the probability of x successes out of n trials is given by

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

(i) $P(1 \text{ head}) = P(x = 1) = {}^{4}C_{1}\,(0.5)^{1}(0.5)^{3} = \mathbf{0.25}$

(ii) $P(\text{at most 3 heads}) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$

$$= {}^{4}C_{0}\,(0.5)^{0}(0.5)^{4} + {}^{4}C_{1}\,(0.5)^{1}(0.5)^{3} + {}^{4}C_{2}\,(0.5)^{2}(0.5)^{2} + {}^{4}C_{3}\,(0.5)(0.5)^{3}$$

$$= (0.5)^{4}(1 + 4 + 6 + 4) = \mathbf{0.9375}$$

(iii) $P(\text{atleast 2 heads}) = 1 - [P(x = 0) + P(x = 1)]$

$$= 1 - [(0.5)^{4} + {}^{4}C_{1}\,(0.5)(0.5)^{3}] = \mathbf{0.6875}$$

---

18. *The probability that a pen manufactured by a factory be defective is 1/10. If 12 such pens are manufactured, what is the probability that **(i)** exactly 2 are defective **(ii)** atleast 2 are defective **(iii)** none of them are defective.*

>> Probability of a defective pen is $p = 1/10 = 0.1$

Probability of a non-defective pen $= q = 1 - p = 1 - 0.1 = 0.9.$

We have $P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$ where we have $n = 12$

(i) Prob. (exactly two defectives) is $P(x = 2)$

$$= {}^{12}C_{2}\,(0.1)^{2}(0.9)^{10} = \mathbf{0.2301}$$

(ii) Prob. (atleast 2 defectives) is $1 - [P(x = 0) + P(x = 1)]$

$$= 1 - [{}^{12}C_{0}\,(0.1)^{0}(0.9)^{12} + {}^{12}C_{1}\,(0.1)^{1}(0.9)^{11}] = \mathbf{0.341}$$

(iii) Prob. (no defective) is $P(x = 0)$

$$= {}^{12}C_{0}\,(0.1)^{0}(0.9)^{12} = (0.9)^{12} = \mathbf{0.2824}$$

---

19. *In a consignment of electric lamps 5 % are defective. If a random sample of 8 lamps are inspected what is the probability that one or more lamps are defective ?*

>> Probability of a defective lamp $= p = 5/100 = 0.05 \quad \therefore \; q = 0.95$

We have $n = 8$ and hence $P(x) = 8_{C_x} (0.05)^x (0.95)^{8-x}$ where x denotes a defective lamp.

Prob. (one or more lamps defective) $= P(1) + P(2) + \cdots + P(8)$

$$= 1 - P(0)$$

$$= 1 - \left[(0.95)^8 \right] = 0.3366$$

Thus the required probability is 0.3366

---

20. *The probability that a person aged 60 years will live upto 70 is 0.65. What is the probability that out of 10 persons aged 60 atleast 7 of them will live upto 70.*

>> Let x be the number of persons aged 60 years living upto 70 years.
For this variate we have by data

$$p = 0.65. \quad \text{Hence} \quad q = 0.35$$

Consider $P(x) = n_{C_x} p^x q^{n-x}$. Here $n = 10$

We have to find $P(x \geq 7)$. That is given by

$$= P(7) + P(8) + P(9) + P(10)$$

$$= 10_{C_7} (0.65)^7 (0.35)^3 + 10_{C_8} (0.65)^8 (0.35)^2 + 10_{C_9} (0.65)^9 (0.35) + (0.65)^{10}$$

But $10_{C_7} = 10_{C_3} = \dfrac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120 \; ; \; 10_{C_8} = 10_{C_2} = \dfrac{10 \cdot 9}{1 \cdot 2} = 45 \; ; 10_{C_9} = 10_{C_1} = 10$

Hence $\mathbf{P(x \geq 7) = 0.5138}$

---

21. *The number of telephone lines busy at an instant of time is a binomial variate with probability 0.1 that a line is busy. If 10 lines are chosen at random, what is the probability that (i) no line is busy (ii) all lines are busy (iii) atleast one line is busy (iv) atmost 2 lines are busy.*

>> Let x denote the number of telephone lines busy. For this variate we have by data,

$$p = 0.1 \; ; \; q = 1 - p = 0.9. \quad \text{Also } n = 10$$

We have $P(x) = n_{C_x} p^x q^{n-x} = 10_{C_x} (0.1)^x (0.9)^{10-x}$

(i) Probability that no line is busy $= P(0) = (0.9)^{10} = \mathbf{0.3487}$

(ii) Probability that all lines are busy $= P(10) = \mathbf{(0.1)^{10}}$

(iii) Probability that atleast one line is busy

$= 1 -$ Probability of no line is busy

$= 1 - P(0) = 1 - 0.3487 = \mathbf{0.6513}$

(iv) Probability that atmost 2 lines are busy

$= P(0) + P(1) + P(2)$

$= (0.9)^{10} + 10_{C_1}(0.1)^1(0.9)^9 + 10_{C_2}(0.1)^2(0.9)^8$

$= \mathbf{0.9298}$

---

22. *In a quiz contest of answering 'Yes' or 'No' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked ? Also find the probability of the same if there are 4 options for a correct answer.*

>> Let x denote the correct answer and we have in the first case,
$p = 1/2$ and $q = 1/2$

$$P(x) = n_{C_x} p^x q^{n-x} = 10_{C_x} (1/2)^x (1/2)^{10-x} = 10_{C_x} \cdot 1/2^{10}$$

We have to find $P(x \geq 6)$

$$P(x \geq 6) = \frac{1}{2^{10}}\left[10_{C_6} + 10_{C_7} + 10_{C_8} + 10_{C_9} + 10_{C_{10}}\right]$$

$$= \frac{1}{2^{10}}(210 + 120 + 45 + 10 + 1) = \frac{386}{1024} = 0.377$$

Thus $\mathbf{P(x \geq 6) = 0.377}$

In the second case when there are 4 options

$$p = 1/4,\ q = 3/4\ ;\ n = 10$$

$$P(x) = 10_{C_x}(1/4)^x(3/4)^{10-x} = \frac{1}{4^{10}}\left[3^{10-x} \cdot 10_{C_x}\right]$$

Hence $P(x \geq 6) = P(6) + P(7) + P(8) + P(9) + P(10)$

$$= \frac{1}{4^{10}}\left[3^4 \cdot 10_{C_6} + 3^3 \cdot 10_{C_7} + 3^2 \cdot 10_{C_8} + 3 \cdot 10_{C_9} + 10_{C_{10}}\right]$$

$$= \frac{1}{4^{10}}\left[81 \times 210 + 27 \times 120 + 9 \times 45 + 3 \times 10 + 1\right] = 0.019$$

Thus $\mathbf{P(x \geq 6) = 0.019}$

---

23. *In sampling a large number of parts manufactured by a company, the mean number of defectives in samples of 20 is 2. Out of 1000 such samples how many would be expected to contain atleast 3 defective parts.*

>> Mean $(\mu) = np = 2$ by data, where $n = 20$

i.e., $20p = 2 \quad \therefore\ p = 1/10 = 0.1$ Hence $q = 1 - p = 0.9$

Let x denote the defective part.

$$P(x) = n_{C_x}\, p^x q^{n-x} = 20_{C_x}\,(0.1)^x (0.9)^{20-x}$$

Probability of atleast 3 defective parts

$$= P(3) + P(4) + \cdots + P(20)$$

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - [(0.9)^{20} + 20_{C_1}(0.1)(0.9)^{19} + 20_{C_2}(0.1)^2 (0.9)^{18}]$$

$$= 0.323$$

Thus the number of defectives in 1000 samples is $1000 \times 0.323 =$ **323**

---

24. *If the mean and standard deviation of the number of correctly answered questions in a test given to 4096 students are 2.5 and $\sqrt{1.875}$. Find an estimate of the number of candidates answering correctly **(i)** 8 or more questions **(ii)** 2 or less **(iii)** 5 questions.*

>> We have mean $(\mu) = np$ and $S.D(\sigma) = \sqrt{npq}$ for a binomial distribution.

By data $np = 2.5$ and $\sqrt{npq} = \sqrt{1.875}$ or $npq = 1.875$

Hence we have $2.5q = 1.875 \quad \therefore\ q = 0.75\ ;\ p = 1 - q = 0.25$

Since $np = 2.5$ we have $(0.25)n = 2.5 \quad \therefore\ n = 10$

Let x denote the number of correctly answered questions.

$$P(x) = n_{C_x}\, p^x q^{n-x} = 10_{C_x}\,(1/4)^x\,(3/4)^{10-x}$$

i.e., $$P(x) = \frac{1}{4^{10}}\left[10_{C_x}\,(3)^{10-x}\right]$$

Since the estimate is needed for 4096 students we have

$$4096\,P(x) = \frac{4096}{4^{10}}[10_{C_x}\,(3)^{10-x}] = \frac{2^{12}}{2^{20}}[10_{C_x}\,(3)^{10-x}]$$

i.e., $$4096\,P(x) = \frac{1}{256}[10_{C_x}(3)^{10-x}] = f(x)\ (say)$$

(i) We have to find $f(8)+f(9)+f(10)$

$$= \frac{1}{256}\,[\,10_{C_8}\cdot 3^2+10_{C_9}\cdot 3+1\,]$$

$$= \frac{1}{256}\,[\,9\cdot 10_{C_2}+3\cdot 10_{C_1}+1\,] = \frac{1}{256}\,(436) = 1.703 \approx 2$$

Number of students correctly answering 8 or more questions is 2.

(ii) We have to find $f(2)+f(1)+f(0)$

$$= \frac{1}{256}\,[\,10_{C_2}\cdot 3^8+10_{C_1}\cdot 3^9+3^{10}\,]$$

$$= \frac{3^8}{256}\,(45+30+9) = \frac{3^8}{256}\,(84) = 2152.8 \approx 2153$$

No. of students correctly answering 2 or less than 2 questions is 2153.

(iii) We have to find $f(5)$

$$= \frac{1}{256}\,[\,10_{C_5}\cdot 3^5\,] = 239.2 \approx 239$$

Number of students correctly answering 5 questions is 239.

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25. *An air line knows that 5 % of the people making reservations on a certain flight will not turn up. Consequently their policy is to sell 52 tickets for a flight that can only hold 50 people. What is the probability that there will be a seat for every passenger who turns up ?*

>> The probability (p) that a passenger will not turn up is

$$p = 0.05 \quad \therefore\ q = 0.95$$

Let x denote the number of passengers who will not turn up.

$$P(x) = n_{C_x}\, p^x q^{n-x} \text{ where } n = 52$$

$$\therefore \quad P(x) = 52_{C_x}\,(0.05)^x\,(0.95)^{52-x}$$

A seat is assured for every passenger who turns up if the number of passengers who fail to turn up is more than or equal to 2.

Hence we have to find $P(x \geq 2)$

$$P(x \geq 2) = 1-[\,P(x<2)\,]$$

$$= 1-[\,P(x=0)+P(x=1)\,]$$

$$= 1-[\,(0.95)^{52}+52\,(0.05)\,(0.95)^{51}\,]$$

$$= 1-0.2595 = 0.7405$$

Probability that a seat is available for every passenger is 0.7405

---

26. *In 800 families with 5 children each how many families would be expected to have* ***(i)*** *3 boys* ***(ii)*** *5 girls* ***(iii)*** *either 2 or 3 boys* ***(iv)*** *atmost 2 girls by assuming probabilities for boys and girls to be equal.*

>> p = probability of having a boy = 1/2

q = probability of having a girl = 1/2

Let x denote the number of boys in the family.

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad \text{where } n = 5$$

i.e., $$P(x) = {}^{5}C_{x}\,(1/2)^{x}(1/2)^{5-x} = \frac{1}{2^{5}}\cdot {}^{5}C_{x} = \frac{{}^{5}C_{x}}{32}$$

Since we have to find the expected number in respect of 800 families we have

$$800\,P(x) = 800\cdot\frac{{}^{5}C_{x}}{32} = 25\cdot{}^{5}C_{x} = f(x)\ \textit{(say)}$$

(i) We have to find $f(3)$

$$f(3) = 25\cdot{}^{5}C_{3} = 25\times 10 = 250$$

Expected number of families with 3 boys is 250.

(ii) We have to find $f(0)$

$$f(0) = 25\cdot{}^{5}C_{0} = 25\times 1 = 25$$

Expected number of families with 5 girls is 25.

(iii) We have to find $f(2)+f(3)$

$$= 25\cdot{}^{5}C_{2} + 25\cdot{}^{5}C_{3} = 50\cdot{}^{5}C_{2} = 50\times 10 = 500$$

Expected number of families with 2 or 3 boys is 500.

(iv) At most 2 girls means that, families can have 5 boys and 0 girls or 4 boys and 1 girl or 3 boys and 2 girls.

Hence we have to find $f(5)+f(4)+f(3)$

$$= 25\cdot{}^{5}C_{5} + 25\cdot{}^{5}C_{4} + 25\cdot{}^{5}C_{3}$$

$$= 25(1+5+10) = 25\times 16 = 400$$

Expected number of families with atmost 2 girls is 400.

27. *Five dice were thrown 96 times and the number of times an odd number actually turned out in the experiment is given. Fit a binomial distribution to this data and calculate the expected frequencies.*

No. of dice showing 1 or 3 or 5	0	1	2	3	4	5
Observed frequency	1	10	24	35	18	8

>> p = probability of getting 1 or 3 or 5 = 3/6 = 1/2 $\therefore$ $q = 1/2$

Here let x denote the number of times an odd number turning out.

$$P(x) = {}^nC_x\, p^x q^{n-x} \text{ where } n = 5$$

Hence $P(x) = {}^5C_x (1/2)^x (1/2)^{5-x} = \dfrac{1}{2^5}\,{}^5C_x$

This is the binomial probability distribution function.

Since 5 dice were thrown 96 times, expected frequencies are obtained from $f(x) = 96\,P(x)$ where $x = 0, 1, 2, 3, 4, 5$

Consider $f(x) = 96 \cdot \dfrac{1}{2^5}\,{}^5C_x = 3 \cdot {}^5C_x$

Hence $f(0) = 3 \cdot {}^5C_0 = 3$; $f(1) = 3 \cdot {}^5C_1 = 3 \times 5 = 15$

$f(2) = 3 \cdot {}^5C_2 = 3 \times 10 = 30$; $f(3) = 3 \cdot {}^5C_3 = 3 \times 10 = 30$

$f(4) = 3 \cdot {}^5C_4 = 3 \times 5 = 15$; $f(5) = 3 \cdot {}^5C_5 = 3 \times 1 = 3$

The expected (theoretical) frequencies are

3, 15, 30, 30, 15, 3

---

28. *4 coins are tossed 100 times and the following results were obtained. Fit a binomial distribution for the data and calculate the theoretical frequencies.*

Number of heads	0	1	2	3	4
Frequency	5	29	36	25	5

>> Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we shall first calculate the mean.

$$\text{Mean } (\mu) = \frac{\sum fx}{\sum f} = \frac{0 + 29 + 72 + 75 + 20}{100} = \frac{196}{100} = 1.96$$

But $\mu = np$ for the binomial distribution. Here $n = 4$

Hence $4p = 1.96$ or $p = 0.49$ $\therefore$ $q = 1-p = 0.51$

Binomial distribution probability function is given by

$$P(x) = {}^{n}C_x\, p^x q^{n-x} = {}^{4}C_x\,(0.49)^x(0.51)^{4-x}$$

Since 4 coins were tossed 100 times, expected (*theoretical*) frequencies are obtained from

$$F(x) = 100\,P(x) = 100\cdot{}^{4}C_x(0.49)^x(0.51)^{4-x}$$

where $x = 0, 1, 2, 3, 4.$

$$F(0) = 100(0.51)^4 = 6.765 \approx 7$$

$$F(1) = 100\cdot{}^{4}C_1(0.49)(0.51)^3 = 400(0.49)(0.51)^3 = 25.999 \approx 26$$

$$F(2) = 100\cdot{}^{4}C_2(0.49)^2(0.51)^2 = 600(0.49)^2(0.51)^2 = 37.47 \approx 37$$

$$F(3) = 100\cdot{}^{4}C_3(0.49)^3(0.51) = 400(0.49)^3(0.51) = 24.0004 \approx 24$$

$$F(4) = 100\cdot{}^{4}C_4(0.49)^4 = 100(0.49)^4 = 5.765 \approx 6$$

Thus the required theoretical frequencies are

7, 26, 37, 24, 6

---

29. *A lot contains 1 % of defective items. What should be the number (n) of items in a random sample so that the probability of finding atleast one defective in it is atleast 0.75 ?*

>> Let p be the probability of a defective item.

By data $p = 1\,\% = 0.01$ $\therefore$ $q = 0.99$

If x denotes a defective item,

$$P(x) = {}^{n}C_x\, p^x q^{n-x} = {}^{n}C_x(0.01)^x(0.99)^{n-x}$$

We need to find n such that the probability of finding atleast one defective is ≥ 0.75.

That is to find n such that

$P(x \geq 1) \geq 0.75$

i.e., $1 - P(x < 1) \geq 0.75$

i.e., $1 - P(0) \geq 0.75$

i.e., $1 - (0.99)^n \geq 0.75$ or $0.25 \geq (0.99)^n$

Equivalently we have $\log(0.25) \geq n \log(0.99)$

or $$n \leq \frac{\log(0.25)}{\log(0.99)} = 137.935$$

Hence the required n is 138.

---

30. *The probability of a shooter hitting a target is 1/3. How many times he should shoot so that the probability of hitting the target atleast once is more than 3/4.*

>> Let p = probability of hitting a target = 1/3. Hence $q = 2/3$

$$P(x) = {}^nC_x\, p^x q^{n-x} = {}^nC_x (1/3)^x (2/3)^{n-x}$$

We have to find n such that

$$P(x \geq 1) > 3/4$$

i.e., $1 - P(x < 1) > 3/4$ or $1 - P(0) > 3/4$

i.e., $1 - (2/3)^n > 3/4$ or $(2/3)^n < 1/4$

We can find n by inspection as we have

$$2/3 \approx 0.67,\ (2/3)^2 \approx 0.44,\ (2/3)^3 \approx 0.3,\ (2/3)^4 \approx 0.2$$

Hence the required n is 4.

---

31. *Show that for a binomial distribution*

$$P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} P(x)$$

>> We have the binomial distribution function,

$$P(x) = {}^nC_x\, p^x q^{n-x} \qquad \ldots(1)$$

$$\therefore \quad P(x+1) = {}^nC_{x+1}\, p^{x+1} q^{n-(x+1)}$$

$$= \frac{n!}{(x+1)!\,[n-(x+1)]!}\, p^{x+1} q^{n-(x+1)}$$

Multiplying and dividing by $(n-x)$ in the RHS we have,

$$P(x+1) = \frac{n!\,(n-x)}{(x+1)\cdot x!\,(n-x)\cdot(n-x-1)!}\, p^{x+1} q^{n-(x+1)}$$

$$= \frac{n-x}{x+1} \cdot \frac{n!}{x!\cdot(n-x)!}\, p \cdot p^x \cdot \frac{q^{n-x}}{q}.$$

$$P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot {}^nC_x \, p^x q^{n-x}$$

By using (1) in the RHS we have,

$$\mathbf{P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \, P(x)}$$

---

32. *Fit a Poisson distribution for the following data and calculate the theoretical frequencies :*

x	0	1	2	3	4
f	122	60	15	2	1

>> We shall first compute the mean (μ) of the given distribution.

$$\mu = \frac{\Sigma f x}{\Sigma f} = \frac{0 + 60 + 30 + 6 + 4}{200} = 0.5$$

We have mean (μ) = m for the Poisson distribution.

The Poisson distribution is

$$P(x) = \frac{m^x e^{-m}}{x!} \text{ and let } f(x) = 200 \cdot P(x)$$

i.e., $f(x) = 200 \frac{(0.5)^x e^{-0.5}}{x!}$ But $e^{-0.5} \approx 0.6065.$

$\therefore \quad f(x) = 121.3 \frac{(0.5)^x}{x!}$

Putting $x = 0, 1, 2, 3, 4$ in $f(x)$ we obtain the theoretical frequencies. They are as follows.

$$121.3 \frac{(0.5)^0}{0!} \approx 121, \; 121.3 \frac{(0.5)^1}{1!} \approx 61,$$

$$121.3 \frac{(0.5)^2}{2!} \approx 15, \; \frac{121.3 \, (0.5)^3}{3!} \approx 3, \; 121.3 \frac{(0.5)^4}{4!} \approx 0$$

Thus the required theoretical frequencies are 121 , 61 , 15 , 3 , 0

---

33. *The number of accidents per day (x) as recorded in a textile industry over a period of 400 days is given. Fit a Poisson distribution for the data and calculate the theoretical frequencies.*

x	0	1	2	3	4	5
f	173	168	37	18	3	1

>> We have for the Poisson distribution :

$$\text{Mean} \quad (\mu) = m = \frac{\sum fx}{\sum f} = \frac{0 + 168 + 74 + 54 + 12 + 5}{400} = 0.7825$$

The Poisson distribution is $P(x) = \dfrac{m^x e^{-m}}{x!}$

Let $\quad f(x) = 400\, P(x)$

$$= 400\ \frac{(0.7825)^x\ e^{-0.7825}}{x!} \qquad \text{But} \quad e^{-0.7825} \approx 0.4573$$

$$\therefore \quad f(x) = 182.9\ \frac{(0.7825)^x}{x!}$$

Theoretical frequencies are got by substituting $x = 0, 1, 2, 3, 4, 5$ in $f(x)$ and they are as follows.

$$(182.9)\,1 \approx 183 \quad ; \quad (182.9)\,(0.7825) \approx 143$$

$$(182.9)\ \frac{(0.7825)^2}{2} \approx 56 \quad ; \quad \frac{(182.9)\,(0.7825)^3}{6} \approx 15$$

$$(182.9)\ \frac{(0.7825)^4}{24} \approx 3 \quad ; \quad \frac{(182.9)\,(0.7825)^5}{120} \approx 0$$

Thus the theoretical frequencies are 183 , 143 , 56 , 15 , 3 , 0

---

34. *In a certain factory turning out razor blades there is a small probability of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing* ***(i)*** *no defective* ***(ii)*** *one defective* ***(iii)*** *two defective blades in a consignment of 10,000 packets.*

>> p = probability of a defective blade $= 1/500 = 0.002$

In a packet of 10, the mean number of defective blades is

$$m = np = 10 \times 0.002 = 0.02$$

Poisson distribution is $P(x) = \dfrac{m^x e^{-m}}{x!} = \dfrac{e^{-0.02}\,(0.02)^x}{x!}$

Let $\quad f(x) = 10{,}000\, P(x)$; Also $e^{-0.02} \approx 0.9802$

$$\therefore \quad f(x) = \frac{9802\,(0.2)^x}{x!}$$

(i) Probability of no defective $= f(0) = \mathbf{9802}$

(ii) Probability of one defective $= f(1) = 9802\ (0.02) \approx \mathbf{196}$

(iii) Probability of two defectives $= f(2) = \dfrac{9802\,(0.02)^2}{2!} \approx \mathbf{2}$

---

35. *The number of accidents in a year to taxi drivers in a city follows a Poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of the drivers with*
(i) no accident in a year
(ii) more than 3 accidents in a year.

>> By data, mean $(\mu) = 3$ and we have for the Poisson distribution $\mu = m = 3$

The Poisson distribution is given by $P(x) = \dfrac{m^x e^{-m}}{x!} = \dfrac{e^{-3} m^x}{x!}$

Let $f(x) = 1000\,P(x)$

i.e., $f(x) = 1000\,\dfrac{3^x e^{-3}}{x!} = 50 \cdot \dfrac{3^x}{x!}$ since $e^{-3} = 0.05$

(i) Number of drivers with no accident in a year $= f(0) = 50 \cdot \dfrac{3^0}{0!} = \mathbf{50}$

(ii) Probability of more than 3 accidents in a year $= 1 - P(x \le 3)$

i.e.,
$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$
$$= 1 - \left[e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right)\right]$$
$$= 1 - [0.05\,(1 + 3 + 4.5 + 4.5)] = 0.35$$

$\therefore$ number of drivers out of 1000 with more than 3 accidents in a year

$= 1000 \times 0.35 = \mathbf{350}$

---

36. *2% of the fuses manufactured by a firm are found to be defective. Find the probability that a box containing 200 fuses contains*
(i) no defective fuses (ii) 3 or more defective fuses.

>> p = probability of a defective fuse $= 2/100 = 0.02$

$\therefore$ mean number of defectives $\mu = m = np = 200 \times 0.02 = 4$

The Poisson distribution is given by $P(x) = \dfrac{m^x e^{-m}}{x!}$

i.e., $P(x) = \dfrac{4^x e^{-4}}{x!}$ But $e^{-4} = 0.0183$

$\therefore \quad P(x) = 0.0183 \cdot \dfrac{4^x}{x!}$

(i) Probability of no defective fuse $= P(0) = \mathbf{0.0183}$

(ii) Probability of 3 or more defective fuses

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - 0.0183\left[1 + \frac{4^1}{1!} + \frac{4^2}{2!}\right]$$

$$= 1 - 0.0183(1 + 4 + 8) \approx \mathbf{0.7621}$$

---

37. *If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000 individuals, more than two will get a bad reaction.*

>> As the probability of occurence (*bad reaction*) is very small, this follows Poisson distribution and we have

$$P(x) = \frac{m^x e^{-m}}{x!}$$

Mean $= m = np = 2000 \times 0.001 = 2$

We have to find $P(x > 2)$

$$P(x > 2) = 1 - P(x \le 2)$$

i.e., $\quad P(x > 2) = 1 - [P(x = 0) + P(x = 1) + P(x = 2)]$

$$= 1 - e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!}\right]$$

$$P(x > 2) = 1 - e^{-2}[1 + 2 + 2] = 1 - 5e^{-2} = \mathbf{0.3222}$$

---

38. *A communication channel receives independent pulses at the rate of 12 pulses per micro second. The probability of transmission error is 0.001 for each micro second. Compute the probabilities of* **(i)** *no error during a micro second* **(ii)** *one error per micro second* **(iii)** *atleast one error per micro second* **(iv)** *two errors* **(v)** *atmost two errors.*

>> Mean number of errors in one micro second

$$\mu = np = m = 12 \times 0.001 = 0.012$$

The Poisson distribution is $P(x) = \dfrac{m^x e^{-m}}{x!}$

i.e., $\quad P(x) = e^{-0.012}\dfrac{(0.012)^x}{x!}$

(i) $P(0) = \mathbf{0.988072}$

(ii) $P(1) = e^{-0.012} \times 0.012 = \mathbf{0.01186}$

(iii) Probability of atleast one error

$$= 1 - P(0) = 1 - e^{-0.012} = \mathbf{0.01193}$$

(iv) $P(2) = e^{-0.012} \dfrac{(0.012)^2}{2!} = \mathbf{0.000071}$

(v) Probability of atmost two errors

$$= P(0) + P(1) + P(2)$$

$$= e^{-0.012}\left[1 + 0.012 + \frac{(0.012)^2}{2}\right] = \mathbf{0.999999714} \approx 1$$

---

39. *A shop has 4 disel generator sets which it hires every day. The demand for a gen set on an average is a poisson variate with value 5/2. Obtain the probability that on a particular day* ***(i)*** *there was no demand*
(ii) *a demand had to be refused.*

>> By data m = mean demand for a generator = 5/2 = 2.5

Poisson distribution is given by $P(x) = \dfrac{m^x e^{-m}}{x!} = e^{-2.5} \cdot \dfrac{(2.5)^x}{x!}$

(i) No demand for a generator. We have to find $P(0)$

$$P(0) = e^{-2.5} = \mathbf{0.082085}$$

(ii) If a demand had to be refused, there should have been a demand for more than 4 generators. We have to find $P(x > 4)$

$$P(x > 4) = 1 - P(x \le 4)$$

$$= 1 - [P(0) + P(1) + P(2) + P(3) + P(4)]$$

$$= 1 - e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$$

Thus $\quad \mathbf{P(x > 4) = 0.108822}$

---

40. *The probability that a news reader commits no mistake in reading the news is $1/e^3$. Find the probability that on a particular news broadcast he commits*
(i) only 2 mistakes (ii) more than 3 mistakes (iii) atmost 3 mistakes.

>> Taking a note of the probability given in the data as $1/e^3 = e^{-3}$ we consider the Poisson distribution

$$P(x) = \frac{m^x e^{-m}}{x!} \text{ where } x \text{ denotes committing a mistake.}$$

By data $P(x=0) = e^{-3}$ and hence we have,

$$e^{-m} = e^{-3} \Rightarrow m = 3 \qquad \therefore \qquad P(x) = e^{-3} \cdot \frac{3^x}{x!}$$

(i) Probability of commiting only two mistakes is $P(2)$

$$P(2) = e^{-3} \cdot \frac{3^2}{2!} = \mathbf{0.22404}$$

(ii) Probability of committing more than 3 mistakes is $P(x > 3)$

$$P(x > 3) = 1 - P(x \le 3)$$

i.e., $P(x > 3) = 1 - [P(0) + P(1) + P(2) + P(3)]$

$$= 1 - e^{-3}\left[1 + 3 + \frac{9}{2} + \frac{27}{6}\right]$$

$$P(x > 3) = 1 - e^{-3}(13) = \mathbf{0.3528}$$

(iii) Probability of committing atmost 3 mistakes

$$= P(0) + P(1) + P(2) + P(3) = e^{-3}(13) = \mathbf{0.6472}$$

---

41. *The probabilities of a Poisson variate taking the values 3 and 4 are equal. Calculate the probabilities of the variate taking the values 0 and 1.*

>> We have $P(x) = \dfrac{m^x e^{-m}}{x!}$ and $P(3) = P(4)$ by data.

$$\therefore \qquad \frac{m^3 e^{-m}}{3!} = \frac{m^4 e^{-4}}{4!}$$

ie., $\dfrac{m^3}{6} = \dfrac{m^4}{24}$ or $6m = 24 \Rightarrow m = 4$

Hence $P(x) = e^{-4} \dfrac{4^x}{x!}$

Now $P(0) = e^{-4} = 1/e^4$ and $P(1) = e^{-4} \cdot 4 = 4/e^4$

Thus the required probabilities are 0.0183 and 0.07326

---

42. *If X follows a Poisson law such that* $P(X = 2) = (2/3)P(X = 1)$, *find* $P(X = 0)$ *and* $P(X = 3)$

>> We have $P(X = x) = \dfrac{m^x e^{-m}}{x!}$ and by using the data we have,

$$\frac{m^2 e^{-m}}{2!} = \frac{2}{3}\,\frac{m e^{-m}}{1!} \text{ or } \frac{m^2}{2} = \frac{2m}{3} \Rightarrow m = \frac{4}{3}$$

Hence $P(X = x) = e^{-4/3} \cdot \dfrac{(4/3)^x}{x!}$

Thus, $P(X = 0) = e^{-4/3} = \mathbf{0.2636}$

$$P(X = 3) = e^{-4/3} \cdot \frac{(4/3)^3}{3!} = e^{-4/3} \cdot \frac{32}{81} = \mathbf{0.10414}$$

---

43. *If x is a Poisson variate such that* $P(x = 2) = 9P(x = 4) + 90P(x = 6)$, *compute the mean and variance of the Poisson distribution.*

>> We have $P(x) = \dfrac{m^x e^{-m}}{x!}$ and by using the data we have,

$$\frac{m^2 e^{-m}}{2!} = 9\frac{m^4 e^{-m}}{4!} + 90\frac{m^6 e^{-m}}{6!}$$

i.e., $\dfrac{m^2}{2} = \dfrac{9}{24} m^4 + \dfrac{90}{720} m^6$

i.e., $\dfrac{m^2}{2} = \dfrac{3}{8} m^4 + \dfrac{1}{8} m^6$ or $1 = \dfrac{3}{4} m^2 + \dfrac{1}{4} m^4$

i.e., $m^4 + 3m^2 - 4 = 0$ or $(m^2 - 1)(m^2 + 4) = 0$

$\therefore$ $m = \pm 1,\ m = \pm 2i$

Neglecting the imaginary value of m and noting that in a Poisson distribution mean and variance are equal to m we have,

mean = variance = 1, taking the positive value.

---

44. *Prove the following recurrence relation for a Poisson distribution.*

$$P(x+1) = \frac{m}{x+1} P(x)$$

>> We have $P(x) = \dfrac{m^x e^{-m}}{x!}$ ...(1)

$$\therefore \quad P(x+1) = \frac{m^{x+1} e^{-m}}{(x+1)!}$$

$$= \frac{m \cdot m^x e^{-m}}{(x+1) \cdot x!}$$

$$P(x+1) = \frac{m}{x+1} \cdot \frac{m^x e^{-m}}{x!} = \frac{m}{x+1} P(x) \text{ by using (1).}$$

Thus $P(x+1) = \dfrac{m}{x+1} P(x)$

7.7 Continuous Probability Distributions

Binomial and Poisson distributions discussed earlier are discrete probability distributions where in the variate can only take integral values.

We have already defined that a random variable which takes noncountable infinite number of values is called a continuous random variable.

If a variate can take any value in an interval, it will give rise to continuous distribution. When a random variable is identified as continuous, we need to consider various questions connected with the probability of the random variable assuming different values. In this context we need a continuous probability function which is defined as follows.

Definition If for every x belonging to the range of a continuous random variable X, we assign a real number $f(x)$ satisfying the conditions

(1) $f(x) \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

then $f(x)$ is called a ***continuous probability function*** or ***probability density function (p.d.f)***

If (a, b) is a subinterval of the range space of X then the probability that x lies in (a, b) is defined to be the integral of $f(x)$ between a and b. That is,

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx \qquad \ldots(1)$$

Cumulative distribution function

If X is a continuous random variable with probability density function $f(x)$ then the function $F(x)$ defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \qquad \ldots (2)$$

is called the ***cumulative distribution function*** (*c.d.f*) of X.

It is evident from (2) that

$$F(x) = P(X \leq x) = P(-\infty < X \leq x) \text{ and } \frac{d}{dx}[F(x)] = f(x)$$

It should be noted that the probability of a continuous random variable taking a particular value is zero whereas the probability that it take values in an interval is a positive quantity.

If r is any real number then

$$P(x \geq r) = \int_{r}^{\infty} f(x)\,dx \qquad \ldots (3)$$

$$P(x < r) = 1 - P(x \geq r)$$

i.e., $$P(x < r) = 1 - \int_{r}^{\infty} f(x)\,dx \qquad \ldots (4)$$

Remark : $\int_{-\infty}^{\infty} f(x)\,dx = 1$ *geometrically means that the area bounded by the curve* $f(x)$ *and the* x *- axis is equal to unity.*

Also $P(a \leq x \leq b)$ *is equal to the area of the region bounded by the curve* $f(x)$, *the* x *- axis and the ordinates* $x = a$ *and* $x = b$.

Mean and Variance

If X is a continuous random variable with probability density function $f(x)$ where $-\infty < x < \infty$, the *mean* (μ) or *expectation* $E(X)$ and the *variance* (σ^2) of X is defined as follows.

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx \qquad \ldots (5)$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx \qquad \ldots(6)$$

We now introduce two continuous probability distributions namely the *exponential distribution* and the *normal distribution.*

7.8 Exponential Distribution

The continuous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x > 0 \\ 0 & \text{otherwise}, \quad \text{where } \alpha > 0 \end{cases}$$

is known as the exponential distribution.

Evidently $f(x) > 0$ and we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \alpha e^{-\alpha x}\,dx = \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^{\infty} = -(0-1) = 1$$

Thus $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$f(x)$ satisfy both the conditions required for a continuous probability function / probability density function.

7.81 Mean and Standard Deviation of the Exponential Distribution

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \alpha e^{-\alpha x}\,dx$$

$$\mu = \alpha \int_{0}^{\infty} x e^{-\alpha x}\,dx$$

Applying Bernoulli's rule of integration by parts we have,

$$\mu = \alpha \left[x \cdot \left(\frac{e^{-\alpha x}}{-\alpha}\right) - 1 \cdot \left(\frac{e^{-\alpha x}}{\alpha^2}\right) \right]_0^{\infty}$$

(Here $x/e^{\alpha x} \to 0$ as $x \to \infty$ by L' Hospital's rule)

$$\mu = \alpha\left[0 - \frac{1}{\alpha^2}(0-1)\right] = \frac{1}{\alpha} \quad ; \mu = \frac{1}{\alpha}$$

Variance $(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$

$$\sigma^2 = \alpha \int_0^{\infty} (x-\mu)^2 e^{-\alpha x}\,dx$$

Applying Bernoulli's rule we have,

$$\sigma^2 = \alpha\left[(x-\mu)^2 \cdot \left(\frac{e^{-\alpha x}}{-\alpha}\right) - 2(x-\mu)\cdot\left(\frac{e^{-\alpha x}}{\alpha^2}\right) + 2\cdot\frac{e^{-\alpha x}}{-\alpha^3}\right]_0^{\infty}$$

$$= \alpha\left[\frac{-1}{\alpha}\left\{0-\mu^2\right\} - \frac{2}{\alpha^2}\left\{0-(-\mu)\right\} - \frac{2}{\alpha^3}\left\{0-1\right\}\right]$$

$$= \alpha\left\{\frac{\mu^2}{\alpha} - \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3}\right\} \text{ But } \mu = \frac{1}{\alpha}$$

$$\therefore \quad \sigma^2 = \alpha\left(\frac{1}{\alpha^3} - \frac{2}{\alpha^3} + \frac{2}{\alpha^3}\right) = \frac{1}{\alpha^2} \quad \text{Hence } \sigma = \frac{1}{\alpha}$$

Thus for the exponential distribution

$$\textbf{Mean } (\mu) = \frac{1}{\alpha} \quad ; \quad \textbf{S.D } (\sigma) = \frac{1}{\alpha}$$

Remark : *Mean = S.D for the exponential distribution.*

7.9 Normal Distribution

The continuous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Putting $t = \frac{x-\mu}{\sqrt{2}\,\sigma}$ or $x = \mu + \sqrt{2}\,\sigma t$, we have $dx = \sqrt{2}\,\sigma\, dt$

t also varies from $-\infty$ to ∞

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\,\sigma\, dt$

i.e., $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{\pi}} \cdot 2 \int_{0}^{\infty} e^{-t^2}\,dt$

But $\int_{0}^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$ by gamma functions.

$\therefore \quad \int_{-\infty}^{\infty} f(x)\,dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$

Thus $f(x)$ represents a probability density function.

Note : *Normal distribution can be derived as a limiting case of the binomial distribution when n is large, neither p nor q is very small. We can derive* $f(x)$ *as* $n \to \infty$ *and* $p = 1/2 = q$

7.91 Mean and Standard Deviation of the Normal Distribution

Mean $(\mu) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Putting $t = \frac{x-\mu}{\sqrt{2}\,\sigma}$ or $x = \mu + \sqrt{2}\,\sigma t$, we have $dx = \sqrt{2}\,\sigma\, dt$

t also varies from $-\infty$ to ∞

$$\text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\,\sigma t)\, e^{-t^2} \sqrt{2}\,\sigma\, dt$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt + \sigma \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t\, e^{-t^2}\,dt$$

$$\text{Mean} = \frac{2\mu}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt + \sigma \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} dt$$

Here the second integral is zero by a standard property since $t e^{-t^2}$ is an odd function.

Hence, mean $= \dfrac{2\mu}{\sqrt{\pi}} \cdot \dfrac{\sqrt{\pi}}{2} + 0 = \mu$

Thus **mean** $= \mu$

Hence we can say that **the mean of the normal distribution is equal to the mean of the given distribution.**

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx$$

Substituting $t = \dfrac{x-\mu}{\sqrt{2}\,\sigma}$ we have as in the earlier case,

$$\text{Variance} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} \sqrt{2}\,\sigma\, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\, 2 \int_0^{\infty} t^2 e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t\,(2t e^{-t^2})\, dt$$

We know that $\int u\,v\,dt = u \int v\,dt - \int \int v\,dt \cdot u'\,dt$

Taking $u = t,\ v = 2t e^{-t^2}$ and noting that $\int v\,dt = -e^{-t^2}$ we now have,

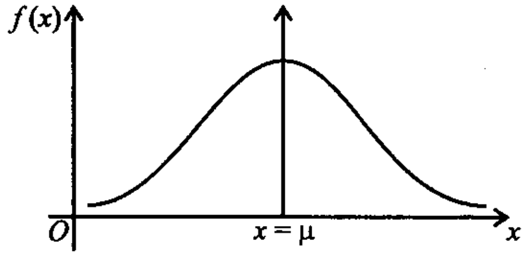
$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \left\{ \left[t(-e^{-t^2}) \right]_0^{\infty} - \int_0^{\infty} -(e^{-t^2}) \cdot 1\, dt \right\}$$

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}}\left\{0 + \int_0^\infty e^{-t^2}\,dt\right\} = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \sigma^2$$

Thus **variance** $= \sigma^2$

Hence we can say that **the variance / S.D of the normal distribution is equal to the variance / S.D of the given distribution.**

Note : *The graph of the probability function $f(x)$ is a bell shaped curve symmetrical about the line $x = \mu$ and is called the normal probability curve. The shape of the curve is as follows.*



The line $x = \mu$ divides the total area under the curve which is equal to 1 into two equal parts. The area to the right as well as to the left of the line $x = \mu$ is 0.5

7.92 Standard Normal Distribution

We have $P(a \le x \le b) = \int_a^b f(x)\,dx$

In the case of normal distribution we have,

$$P(a \le x \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/2\sigma^2}\,dx \qquad \ldots (1)$$

The integral in the RHS of this equation cannot be evaluated by known methods of integration and we have to employ the technique of numerical integration which becomes tedious. Hence we think of standardization and the same is as follows.

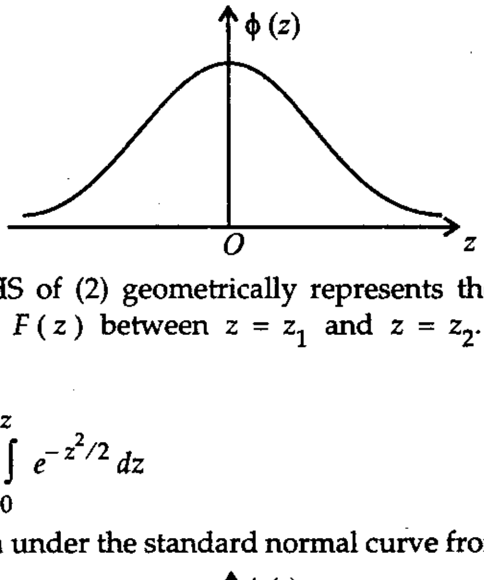
Putting $z = \frac{x-\mu}{\sigma}$ or $x = \mu + \sigma z$ we have $dx = \sigma\,dz$

Let $z_1 = \frac{a-\mu}{\sigma}$ and $z_2 = \frac{b-\mu}{\sigma}$ be the values of z corresponding to $x = a$ and $x = b$. The integral in (1) assumes the following form.

$$P(a \le x \le b) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2} dz$$

Hence, $P(a \le x \le b) = P(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2} dz$ ...(2)

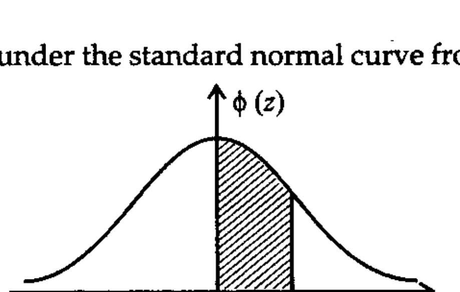
If $F(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ (*standard normal probability density function*), it may be observed that this is same as the *p.d.f* of the normal distribution with $\mu = 0$ and $\sigma = 1$. *Thus we can say that the normal probability density function with* $\mu = 0$ *and* $\sigma = 1$ *is the standard normal probability density function.* $z = \frac{x-\mu}{\sigma}$ *is called the standard normal variate (S.N.V) and* $F(z)$ *is called the standard normal curve which is symmetrical about the line* $z = 0$. *The curve is as follows.*



The integral in the RHS of (2) geometrically represents the area bounded by the standard normal curve $F(z)$ between $z = z_1$ and $z = z_2$. Further in particular if $z_1 = 0$ we have

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-z^2/2} dz$$

This represents the area under the standard normal curve from $z = 0$ to z.



$\phi(z)$ also denoted by $A(z)$ represents the area (*shaded portion*) as shown in the figure. Since the total area is 1, the area on either side of $z = 0$ is 0.5.

Tabulated values which gives the area for different positive values of z are available and this helps us in practical problems. The procedure for using the table will be discussed later.

Table is given at the end of the book

WORKED PROBLEMS

45. *Find which of the following functions is a probability density function*

(i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0 & otherwise \end{cases}$

(ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 1 \\ 0 & otherwise \end{cases}$

(iii) $f_3(x) = \begin{cases} |x|, & |x| \le 1 \\ 0 & otherwise \end{cases}$

(iv) $f_4(x) = \begin{cases} 2x, & 0 < x \le 1 \\ 4-4x, & 1 < x < 2 \\ 0 & otherwise \end{cases}$

>> Conditions for a p.d.f are $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(i) Clearly $f(x) \ge 0$

$$\int_{-\infty}^{\infty} f_1(x) = \int_0^1 f_1(x)\,dx = \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1$$

$\therefore$ **$f_1(x)$ is a p.d.f**

(ii) The given function can be written in the form.

$$f_2(x) = \begin{cases} 2x, & -1 < x < 0 \\ 2x, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In $-1 < x < 0$, $f_2(x) = 2x$ is less than zero.

Further $\int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-1}^{1} 2x\,dx = \left[x^2\right]_{-1}^{1} = 0$

Both the conditions are not satisfied.

$\therefore$ **$f_2(x)$ is not a p.d.f**

(iii) Evidently $f_3(x) = |x| \geq 0$

$$\int_{-\infty}^{\infty} f_3(x) = \int_{-1}^{1} f_3(x)\,dx = \int_{-1}^{1} |x|\,dx$$

But $|x| = \begin{cases} -x, & -1 < x < 0 \\ +x, & 0 < x < 1 \end{cases}$

$\therefore$ $$\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore$ **$f_3(x)$ is a p.d.f**

(iv) The given function

$f_4(x) = 2x > 0$ in $0 < x \leq 1$

But $f_4(x) = 4 - 4x$ is negative in $1 < x < 2$

The first condition is not satisfied.

$\therefore$ **$f_4(x)$ is not a p.d.f**

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46. *Find the value of c such that* $f(x) = \begin{cases} \frac{x}{6} + c, & 0 \leq x \leq 3 \\ 0 & elsewheree \end{cases}$ *is a p.d.f.*

Also find $P(1 \leq x \leq 2)$

>> $f(x) \geq 0$ *if* $c \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $$\int_0^3 \left(\frac{x}{6} + c\right) dx = 1$$

i.e., $$\left[\frac{x^2}{12} + cx\right]_0^3 = 1.$$

i.e., $$\frac{3}{4} + 3c = 1 \quad \therefore \quad c = \frac{1}{12}$$

Now $P(1 \le x \le 2) = \int_1^2 f(x)\,dx$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}[(4+2)-(1+1)] = \frac{1}{3}$$

Thus $P(1 \le x \le 2) = 1/3$

---

47. *Find the constant k such that* $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0 & otherwise \end{cases}$ *is a p.d.f Also compute*
(i) $P(1 < x < 2)$ *(ii)* $P(x \le 1)$ *(iii)* $P(x > 1)$ *(iv) Mean (v) Variance*

>> $f(x) \ge 0$ if $k \ge 0$. Also we must have, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_0^3 kx^2\,dx = 1$

i.e, $\left[\frac{kx^3}{3}\right]_0^3 = 1$ or $9k = 1$ $\therefore k = \frac{1}{9}$

(i) $P(1 < x < 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_1^2 = \frac{7}{27}$

(ii) $P(x \le 1) = \int_0^1 f(x)\,dx = \int_0^1 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_0^1 = \frac{1}{27}$

(iii) $P(x > 1) = \int_1^3 f(x)\,dx = \int_1^3 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_1^3 = \frac{26}{27}$

(iv) Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^3 x \cdot \frac{x^2}{9}\,dx = \left[\frac{x^4}{36}\right]_0^3 = \frac{81}{36} = \frac{9}{4}$

(v) Variance $= V = \int_{-\infty}^{\infty} x^2 f(x)\,dx - (\mu)^2$

$$= \int_0^3 x^2 \cdot \frac{x^2}{9}\,dx - \left(\frac{9}{4}\right)^2$$

$$V = \left[\frac{x^5}{45}\right]_0^3 - \frac{81}{16} = \frac{81}{15} - \frac{81}{16} = \frac{81}{240} = \frac{27}{80}$$

---

48. *Find k such that* $f(x) = \begin{cases} k\,x\,e^{-x}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ *is a p.d.f. Find the mean.*

>> $f(x) \geq 0$ if $k \geq 0$. Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_0^1 k\,x\,e^{-x}\,dx = 1$

Applying Bernoulli's rule we have,

$$k\left[x(-e^{-x}) - (1)(e^{-x})\right]_0^1 = 1$$

i.e., $k\left[-\frac{1}{e} - \left(\frac{1}{e} - 1\right)\right] = 1,$

i.e, $k\left(1 - \frac{2}{e}\right) = 1$

$\therefore$ $k = \frac{e}{e-2}$

Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 x \cdot \frac{e}{e-2}\,x\,e^{-x}\,dx$

$$= \frac{e}{e-2}\int_0^1 x^2 e^{-x}\,dx$$

$$= \frac{e}{e-2}\left[x^2(-e^{-x}) - (2x)(e^{-x}) + 2(-e^{-x})\right]_0^1$$

$$\mu = \frac{e}{e-2}\left[-\frac{1}{e}-\frac{2}{e}-2\left(\frac{1}{e}-1\right)\right]$$

$$\mu = \frac{e}{e-2}\left[2-\frac{5}{e}\right] = \frac{2e-5}{e-2}$$

---

49. *Is the following function a density function ?*

$$f(x) = e^{-x}, \quad x \ge 0$$
$$= 0, \quad x < 0$$

If so, determine the probability that the variate having this density will fall in the interval $(1, 2)$.

>> We observe $f(x) \ge 0$. Also we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\infty} = -(0-1) = 1$$

Hence $f(x)$ is a probability density function.

The probability that the variate having this density will fall in the interval (1,2) is to compute $P(1 < x < 2)$

$$\therefore \quad P(1 < x < 2) = \int_{1}^{2} f(x)\,dx = -\left[e^{-x}\right]_1^2 = -(e^{-2} - e^{-1})$$

Thus $P(1 < x < 2) = (1/e - 1/e^2) =$ **0.2325**

---

50. *A random variable x has the following density function*

$$P(x) = \begin{cases} kx^2, & -3 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and find ***(i)*** $P(1 \le x \le 2)$ ***(ii)*** $P(x \le 2)$ ***(iii)*** $P(x > 1)$

>> $P(x) \ge 0$ if $k \ge 0$ and also we must have $\int_{-\infty}^{\infty} P(x)\,dx = 1$

That is, $\int_{-3}^{3} k x^2 \, dx = 1$

or $\left[\frac{k x^3}{3} \right]_{-3}^{3} = 1 \qquad \therefore \quad k = \frac{1}{18}$

(i) $P(1 \le x \le 2) = \int_{1}^{2} \frac{x^2}{18} \, dx = \left[\frac{x^3}{54} \right]_{1}^{2} = \frac{1}{54}(8-1) = \frac{7}{54}$

(ii) $P(x \le 2) = \int_{-3}^{2} \frac{1}{18} x^2 \, dx = \frac{1}{18} \left[\frac{x^3}{3} \right]_{-3}^{2} = \frac{1}{54}(8+27) = \frac{35}{54}$

(iii) $P(x > 1) = \int_{1}^{3} \frac{1}{18} x^2 \, dx = \frac{1}{18} \left[\frac{x^3}{3} \right]_{1}^{3} = \frac{1}{54}(27-1) = \frac{26}{54} = \frac{13}{27}$

---

51. *Find the c.d.f for the following p.d.f of a random variable x.*

(i) $f(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$

(ii) $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & 0 < x < \infty \\ 0 & otherwise \end{cases}$

(iii) Exponential distribution

>> If $f(x)$ is the p.d.f then the c.d.f $= F(x) = \int_{-\infty}^{x} f(x) \, dx.$

(i) $F(x) = \int_{-\infty}^{x} f(x) \, dx = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{x} f(x) \, dx$

$= 0 + \int_{0}^{x} f(x) \, dx$

$\therefore \quad F(x) = \int_{0}^{x} (6x - 6x^2) \, dx = \left[3x^2 - 2x^3 \right]_{0}^{x} = 3x^2 - 2x^3$

$\therefore$ **c.d.f $= 3x^2 - 2x^3$** if $0 \le x \le 1$

(ii) $F(x) = \int_0^x \frac{x}{4} e^{-x/2}\, dx$

$$= \frac{1}{4}\left[x \cdot \frac{e^{-x/2}}{-1/2} - 1 \cdot \frac{e^{-x/2}}{1/4} \right]_0^x$$

$$= \frac{1}{4}\left[-2\,(x e^{-x/2} - 0) - 4\,(e^{-x/2} - 1) \right]$$

$\therefore$ **c.d.f** $= 1 - e^{-x/2} - (x/2)\, e^{-x/2}$, **if** $0 < x < \infty$

(iii) $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$

is the p.d.f of the exponential distribution.

$$F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_0^x \alpha e^{-\alpha x}\, dx = -\left[e^{-\alpha x} \right]_0^x = 1 - e^{-\alpha x}$$

$\therefore$ **c.d.f** $= 1 - e^{-\alpha x}$ **if** $0 < x < \infty$

---

52. *A continuous random variable has the distribution function*

$$F(x) = \begin{cases} 0\,, & x \le 1 \\ c\,(x-1)^4, & 1 \le x \le 3 \\ 1, & x > 3 \end{cases} \quad \textit{Find c and also the p.d.f}$$

>> We know that the p.d.f $f(x) = \frac{d}{dx}\,[F(x)]$

$$\therefore \quad f(x) = \begin{cases} 0, & x \le 1 \\ 4c\,(x-1)^3, & 1 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

$f(x) \ge 0$ for $c \ge 0$ and we must have $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Hence we must have $\int_1^3 4c\,(x-1)^3\, dx = 1$

That is, $\left[c\,(x-1)^4 \right]_1^3 = 1$, ie., $16c = 1$ $\quad \therefore \; c = 1/16$

Thus the p.d.f $f(x) = (x-1)^3/4$ **where** $1 \leq x \leq 3$.

---

53. *Find k so that the following function can serve as a probability density function of a random variable.*

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ k\,x\,e^{-4x^2} & \text{for } x > 0 \end{cases}$$

>> We must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_{0}^{\infty} k\,x\,e^{-4x^2}\,dx = 1$

Putting $4x^2 = t$, we have $8x\,dx = dt$; t also varies from 0 to ∞.

$\therefore \quad \int_{t=0}^{\infty} k\,e^{-t}\,\frac{dt}{8} = 1$

i.e., $\frac{k}{8}\left[-e^{-t}\right]_0^{\infty} = 1$ or $\frac{k}{8} = 1 \quad \therefore \quad$ **$k = 8$**

---

54. *A random variable x has the density function* $f(x) = k/1+x^2$, $-\infty < x < \infty$ *Determine k and hence evaluate* ***(i)*** $P(x \geq 0)$ ***(ii)*** $P(0 < x < 1)$

>> We must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1.$

Since the integrand is even we have,

$$2\int_0^{\infty} \frac{k}{1+x^2}\,dx = 1$$

i.e., $2k\left[\tan^{-1}x\right]_0^{\infty} = 1$

i.e., $2k[\pi/2 - 0] = 1 \quad \therefore \quad \mathbf{k = 1/\pi}$

Now $P(x \geq 0) = \int_0^\infty f(x)\,dx$

$$= \frac{1}{\pi}\int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^\infty = \frac{1}{\pi}\left[\frac{\pi}{2}\right] = \frac{1}{2}$$

Thus $\mathbf{P(x \geq 0) = 1/2}$

Also $P(0 < x < 1) = \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx$

$$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^1 = \frac{1}{\pi}\left[\frac{\pi}{4}\right] = \frac{1}{4}$$

Thus $\mathbf{P(0 < x < 1) = 1/4}$

---

55. *The time t years required to complete a software project has p.d.f of the form*

$$f(t) = \begin{cases} kt(1-t), & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find k and also the probability that the project will be completed in less than 4 months.

>> We must have $\int_{-\infty}^{\infty} f(t)\,dt = 1$

i.e., $\int_0^1 kt(1-t)\,dt = 1$

i.e., $k\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = 1$ or $k\left[\frac{1}{6}\right] = 1 \quad \therefore \quad \mathbf{k = 6}$

Probability that the project will be completed in 4 months is equivalent to find $P(0 < t < 1/3)$ since t is in years.

$$\therefore \quad P(0 < t < 1/3) = \int_0^{1/3} 6t(1-t)\,dt$$

$$= \left[3t^2 - 2t^3\right]_0^{1/3} = \left[\frac{1}{3} - \frac{2}{27}\right] = \frac{7}{27}$$

Thus the required probability is 7/27

---

56. *The kilometre run (in thousands of k.ms) without any sort of problem in respect of a certain vehicle is a random variable having p.d.f*

$$f(x) = \begin{cases} \frac{1}{40} e^{-x/40}, & x \geq 0 \\ 0, & x \leq 0 \end{cases}$$

Find the probability that the vehicle is trouble free

(i) atleast for 25000 k.ms ***(ii)*** *atmost for 25000 k.ms* ***(iii)*** *between 16000 to 32000 k.ms*

>> Here x is the random variable representing kilometre in multiples of 1000 regarding trouble free run by the vehicle.

(i) To find $P(x \geq 25)$

$$P(x \geq 25) = 1 - P(x < 25)$$

$$= 1 - \int_0^{25} \frac{1}{40} e^{-x/40}\,dx$$

$$= 1 + \left[e^{-x/40}\right]_0^{25} = 1 + \left[e^{-5/8} - 1\right] = e^{-5/8}$$

$$\therefore \quad \mathbf{P(x \geq 25) = 0.5353}$$

(ii) To find $P(x \leq 25)$

$$P(x \leq 25) = \int_0^{25} \frac{1}{40} e^{-x/40}\,dx$$

$$= \left[-e^{-x/40}\right]_0^{25} = -e^{-5/8} + 1$$

$$\therefore \quad \mathbf{P(x \leq 25) = 0.4647}$$

(iii) To find $P(16 \le x \le 32)$

$$P(16 \le x \le 32) = \int_{16}^{32} \frac{1}{40} e^{-x/40} dx$$

$$= \left[-e^{-x/40} \right]_{16}^{32} = -e^{-4/5} + e^{-2/5}$$

$\therefore$ $\mathbf{P(16 \le x \le 32) = 0.221}$

---

57. *If x is an exponential variate with mean 3 find (i)* $P(x > 1)$ *(ii)* $P(x < 3)$

>> The p.d.f of the exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

The mean of this distribution is given by $1/\alpha$

By data, mean $= 1/\alpha = 3 \ \therefore \ \alpha = 1/3$

$$\text{Hence } f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(x > 1) = 1 - P(x \le 1)$

$$= 1 - \int_0^1 f(x)\, dx$$

$$= 1 - \int_0^1 \frac{1}{3} e^{-x/3} dx = 1 + \left[e^{-x/3} \right]_0^1 = e^{-1/3}$$

$\therefore$ $\mathbf{P(x > 1) = e^{-1/3} = 0.7165}$

(ii) $P(x < 3) = \int_0^3 f(x)\, dx$

$$= \int_0^3 \frac{1}{3} e^{-x/3} dx$$

$$= -\left[e^{-x/3} \right]_0^3 = 1 - \frac{1}{e} = 0.6321$$

$\therefore$ $\mathbf{P(x < 3) = 0.6321}$

---

58. *If x is an exponential variate with mean 5, evaluate.*

(i) $P(0 < x < 1)$ *(ii)* $P(-\infty < x < 10)$ *(iii)* $P(x \le 0 \text{ or } x \ge 1)$

>> $f(x) = \alpha e^{-\alpha x},\ 0 < x < \infty$; Mean $= \dfrac{1}{\alpha} = 5 \ \therefore\ \alpha = \dfrac{1}{5}$

Hence $f(x) = \dfrac{1}{5} e^{-x/5},\ 0 < x < \infty$

(i) $$P(0 < x < 1) = \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{1}{5} e^{-x/5}\,dx = -\left[e^{-x/5}\right]_0^1$$

$$P(0 < x < 1) = 1 - e^{-1/5} = 1 - e^{-0.2} = 0.1813$$

$\therefore$ $\mathbf{P(0 < x < 1) = 0.1813}$

(ii) $$P(-\infty < x < 10) = \int_{-\infty}^{0} f(x)\,dx + \int_0^{10} f(x)\,dx$$

$$= \int_0^{10} \frac{1}{5} e^{-x/5}\,dx = -\left[e^{-x/5}\right]_0^{10} = 1 - (1/e^2) = 0.8647$$

$\therefore$ $\mathbf{P(-\infty < x < 10) = 0.8647}$

(iii) $$P(x \le 0 \text{ or } x \ge 1) = P(x \le 0) + P(x \ge 1)$$

$$= 0 + \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{5} e^{-x/5}\,dx$$

$$P(x \le 0 \text{ or } x \ge 1) = -\left[e^{-x/5}\right]_1^{\infty} = -(0 - e^{-0.2}) = e^{-0.2}$$

$\therefore$ $\mathbf{P(x \le 0 \text{ or } x \ge 1) = 0.8187}$

---

59. *The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth* ***(i)*** *ends less than 5 minutes* ***(ii)*** *between 5 and 10 minutes.*

>> We have $f(x) = \alpha\, e^{-\alpha x},\ x > 0$; Mean $= 1/\alpha$

By data $1/\alpha = 5 \quad \therefore \quad \alpha = 1/5$

Hence $f(x) = \frac{1}{5} e^{-x/5}$ is the p.d.f.

(i) $$P(x < 5) = \int_0^5 f(x)\,dx$$

$$= \int_0^5 \frac{1}{5} e^{-x/5} dx = -\left[e^{-x/5}\right]_0^5$$

$$= 1 - (1/e) = 0.6321$$

$\therefore$ $\mathbf{P(x < 5) = 0.6321}$

(ii) $$P(5 < x < 10) = \int_5^{10} f(x)\,dx = \int_5^{10} \frac{1}{5} e^{-x/5}\,dx$$

$$P(5 < x < 10) = -\left[e^{-x/5}\right]_5^{10}$$

$$= (1/e) - (1/e^2) = 0.2325$$

$\therefore$ $\mathbf{P(5 < x < 10) = 0.2325}$

---

60. *The sales per day in a shop is exponentially distributed with the average sale amounting to Rs.100 and net profit is 8%. Find the probability that the net profit exceeds Rs. 30 on two consecutive days.*

>> Let x be the random variable of the sale in the shop. Since x is an exponential variate the p.d.f $f(x) = \alpha e^{-\alpha x},\ x > 0$

$$\text{Mean} = 1/\alpha = 100 \ \therefore\ \alpha = 1/100 = 0.01$$

Hence $f(x) = 0.01\ e^{-0.01x},\ x > 0$

Let A be the amount for which profit is 8%

$$\Rightarrow A \cdot \frac{8}{100} = 30 \ \therefore\ A = 375$$

Probability of profit exceeding Rs.30 is equal to

$$1 - \text{Prob (profit} \le \text{Rs.30)}$$

$$= 1 - \text{Prob (sales} \le \text{Rs.375)}$$

$$= 1 - \int_0^{375} (0.01) e^{-0.01x} = 1 + \left[e^{-0.01x}\right]_0^{375} = e^{-3.75}$$

The probability that profit exceeds Rs. 30 on a single day is $e^{-3.75}$

Thus the probability that it repeats on the following day is

$$e^{-3.75} \cdot e^{-3.75} = e^{-7.5} = \mathbf{0.00055}$$

---

61. *In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for :*

(i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 and 12 minutes

>> The p.d.f of the exponential distribution is given by

$$f(x) = \alpha e^{-\alpha x},\ x > 0 \text{ and the mean } = 1/\alpha$$

By data $1/\alpha = 5 \therefore \alpha = 1/5$ and hence $f(x) = \frac{1}{5} e^{-x/5}$

(i) $P(x \geq 10) = \int_{10}^{\infty} \frac{1}{5} \cdot e^{-x/5}\, dx = -\left[e^{-x/5}\right]_{10}^{\infty}$

$\therefore$ $P(x \geq 10) = -(0 - e^{-2}) = e^{-2} = \mathbf{0.1353}$

(ii) $P(x < 10) = \int_{0}^{10} \frac{1}{5} \cdot e^{-x/5}\, dx = -\left[e^{-x/5}\right]_{0}^{10}$

$\therefore$ $P(x < 10) = -(e^{-2} - 1) = 1 - e^{-2} = \mathbf{0.8647}$

(iii) $P(10 < x < 12) = \int_{10}^{12} \frac{1}{5} \cdot e^{-x/5}\, dx = -\left[e^{-x/5}\right]_{10}^{12}$

$\therefore$ $P(10 < x < 12) = -(e^{-12/5} - e^{-2}) = \mathbf{0.0446}$

Worked Problems on Normal Distribution

We have said that $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-z^2/2}\, dz$

represents the area under the standard normal curve from 0 to z. Tabulated values which gives the area for different positive values of z is readily available. Such a table analogous to the format of a logarithmic table is called *normal probability table.* We present a few illustrations geometrically, theoretically and write the value by making use of the table (*given at the end of the book*). We also bear in mind the following established results.

(1) $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$ (2) $\int_{-\infty}^{0} \phi(z)\,dz = \int_{0}^{\infty} \phi(z)\,dz = \frac{1}{2}$

These results in the equivalent form are as follows.

(1) $P(-\infty \le z \le \infty) = 1$ (2) $P(-\infty \le z \le 0) = 1/2$

(3) $P(0 \le z \le \infty)$ or $P(z \ge 0) = 1/2$

Also $P(-\infty < z < z_1) = P(-\infty < z \le 0) + P(0 \le z < z_1)$

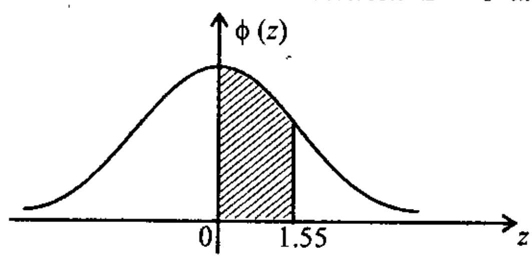
i.e., $P(z < z_1) = 0.5 + \phi(z_1)$ ...(4)

Also $P(z > z_2) = P(z \ge 0) - P(0 \le z < z_2)$

i.e., $P(z > z_2) = 0.5 - \phi(z_2)$ ...(5)

Illustration - 1

To find the area under the standard normal curve between $z = 0$ *and* 1.55



Theoretically the area $= \frac{1}{\sqrt{2\pi}} \int_{0}^{1.55} e^{-z^2/2}\,dz = \phi(1.55)$

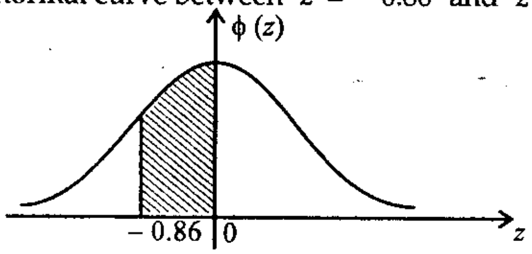
Referring to the table we move vertically down along the column of z to reach 1.5 and then move horizontally along this line to the value 5 (*regarded as .05*) to intersect with the numerical figure 0.4394

Hence $\phi(1.55) = 0.4394$

Equivalently we have $P(0 \le z \le 1.55) = \phi(1.55) = 0.4394$

Illustration - 2

Area of the standard normal curve between $z = -0.86$ and $z = 0$



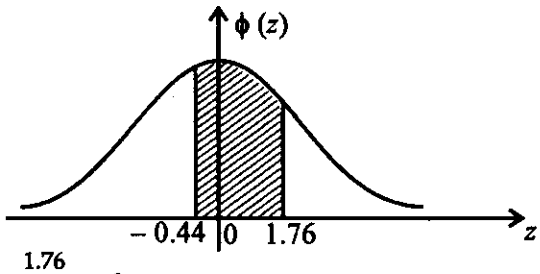
$$\text{Area} = \frac{1}{\sqrt{2\pi}} \int_{-0.86}^{0} e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{0.86} e^{-z^2/2}\, dz \quad \text{by symmetry.}$$

$\therefore$ the required area $= \phi\,(0.86) = 0.3051$

Equivalently $P(-0.86 \le z \le 0) = 0.3051$

Illustration - 3

Area of the standard normal curve between $z = -0.44$ and $z = 1.76$



$$\text{Area} = \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{1.76} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{0} e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{1.76} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{0.44} e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{1.76} e^{-z^2/2}\, dz$$

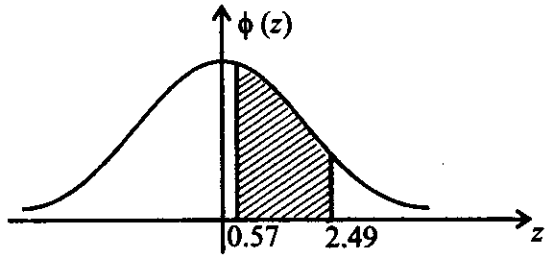
$= \phi\,(0.44) + \phi\,(1.76)$

$= 0.1700 + 0.4608 = 0.6308$

Equivalently $P(-0.44 \le z \le 1.76) = 0.6308$

Illustration - 4

Area of the standard normal curve between $z = 0.57$ to $z = 2.49$



Required area = (Area between $z = 0$ to 2.49) – (Area between $z = 0$ to 0.57)

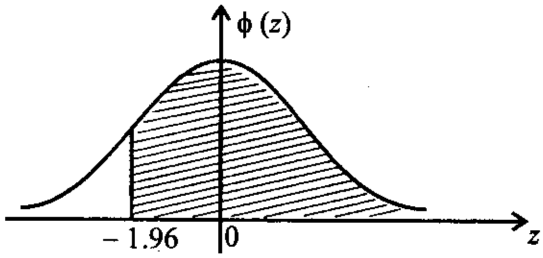
Required area $= \phi\ (2.49) - \phi\ (0.57)$

$= 0.4936 - 0.2157 = 0.2779$

Equivalently $P(0.57 \leq z \leq 2.49) = 0.2779$

Illustration - 5

Area of the standard normal curve to the right of $z = -1.96$



Required area = (Area between $z = -1.96$ to 0) + (Area to the right of $z = 0$)

= (Area between $z = 0$ and 1.96) + 0.5, by symmetry.

$= \phi\ (1.96) + 0.5 = 0.4750 + 0.5 = 0.9750$

Equivalently $P(z \geq -1.96) = 0.975$

---

62. *Evaluate the following probabilities with the help of normal probability tables.*

(i) $P(z \geq 0.85)$ *(ii)* $P(-1.64 \leq z \leq -0.88)$

(iii) $P(z \leq -2.43)$ *(iv)* $P(|z| \leq 1.94)$

>> **(i)** $P(z \geq 0.85) = P(z \geq 0) - P(z \leq 0.85)$

$= 0.5 - \phi\ (0.85)$

$= 0.5 - 0.3023 = 0.1977$

$\therefore$ $\mathbf{P(z \geq 0.85) = 0.1977}$

(ii) $P(-1.64 \leq z \leq -0.88)$

By symmetry, $P(-1.64 \leq z \leq -0.88) = P(0.88 \leq z \leq 1.64)$

$\therefore$ $P(-1.64 \leq z \leq -0.88) = P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88)$

$= \phi(1.64) - \phi\ (0.88)$

$= 0.4495 - 0.3106 = 0.1389$

$\therefore$ $\mathbf{P(-1.64 \leq z \leq -0.88) = 0.1389}$

Remark : *In this case, using the concept of symmetry we can directly write* $\phi\ (1.64) - \phi\ (0.88)$

(iii) $P(z \le -2.43)$

$$P(z \le -2.43) = P(z \ge 2.43)$$
$$= P(z \ge 0) - P(z \le 2.43)$$
$$= 0.5 - \phi(2.43)$$
$$= 0.5 - 0.4925 = 0.0075$$

$\therefore$ $\mathbf{P(z \le -2.43) = 0.0075}$

(iv) $P(|z| \le 1.94) = P(-1.94 \le z \le 1.94)$
$$= 2P(0 \le z \le 1.94)$$
$$= 2\,\phi(1.94) = 2(0.4738) = 0.9476$$

$\therefore$ $\mathbf{P(|z| \le 1.94) = 0.9476}$

---

63. *If x is a normal variate with mean* 30 *and standard deviation* 5 *find the probability that*

(i) $26 \le x \le 40$ *(ii)* $x \ge 45$

>> We have standard normal variate (s.n.v) $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-30}{5}$

(i) To find $P(26 \le x \le 40)$

If $x = 26$, $z = -0.8$; If $x = 40$, $z = 2$

Hence we need to find $P(-0.8 \le z \le 2)$

$$P(-0.8 \le z \le 2) = P(-0.8 \le z \le 0) + P(0 \le z \le 2)$$
$$= P(0 \le z \le 0.8) + P(0 \le z \le 2)$$
$$= \phi(0.8) + \phi(2)$$
$$= 0.2881 + 0.4772 = 0.7653$$

$\therefore$ $\mathbf{P(26 \le x \le 40) = 0.7653}$

(ii) To find $P(x \ge 45)$

If $x = 45$, $z = 3$ and hence we have to find $P(z \ge 3)$

$$P(z \ge 3) = P(z \ge 0) - P(z \le 3)$$
$$= 0.5 - \phi(3)$$
$$= 0.5 - 0.4987 = 0.0013$$

$\therefore$ $\mathbf{P(x \ge 45) = 0.0013}$

---

64. *If x is normally distributed with mean* 12 *and S.D* 4, *find the following.*

(i) $P(x \geq 20)$ *(ii)* $P(x \leq 20)$

>> We have s.n.v $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-12}{4}$

If $x = 20, z = 2$

We have to find $P(z \geq 2)$ and $P(z \leq 2)$

Now $P(z \geq 2) = P(z > 0) - P(z \leq 2)$

$= 0.5 - \phi(2)$

$= 0.5 - 0.4772 = 0.0228$

Also $P(z \leq 2) = P(-\infty \leq z \leq 0) + P(0 \leq z \leq 2)$

$= 0.5 + \phi(2)$

$= 0.5 + 0.4772 = 0.9772$

Thus $\mathbf{P(x \geq 20) = 0.0228}$ and $\mathbf{P(x \leq 20) = 0.9772}$

---

65. *The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be*
(i) less than 65 *(ii) more than 75* *(iii) between 65 and 75*

>> Let x represent the marks of students.

By data $\mu = 70, \sigma = 5$ Hence s.n.v $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-70}{5}$

(i) If $x = 65, z = -1$ and we have to find $P(z < -1)$

$P(z < -1) = P(z > 1)$

$= P(z \geq 0) - P(0 < z < 1)$

$= 0.5 - \phi(1) = 0.5 - 0.3413 = 0.1587$

$\therefore$ **number of students scoring less than 65 marks**

$= 1000 \times 0.1587 = 158.7 \approx$ **159**

(ii) If $x = 75, z = 1$ and we have to find $P(z > 1)$

$P(z > 1) = P(z \geq 0) - P(0 < z < 1)$

$= 0.5 - \phi(1) = 0.5 - 0.3413 = 0.1587$

$\therefore$ **number of students scoring more than 75 marks**

$= 1000 \times 0.1587 = 158.7 \approx$ **159**

(iii) We have to find $P(-1 < z < 1)$

$$P(-1 < z < 1) = 2P(0 < z < 1)$$
$$= 2\,\phi\,(1) = 2(0.3413) = 0.6826$$

$\therefore$ **number of students scoring marks between 65 and 75**

$= 1000 \times 0.6826 = 682.6 \approx$ **683**

---

66. *In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours. If a firm purchases 2500 bulbs find the number of bulbs that are likely to last for* ***(i)*** *more than 2100 hours.* ***(ii)*** *less than 1950 hours* ***(iii)*** *between 1900 to 2100 hours*

>> By data $\mu = 2000,\ \sigma = 60$

We have s.n.v $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-2000}{60}$

(i) To find $P(x > 2100)$

If $x = 2100,\ z = 100/60 \approx 1.67$

$$P(x > 2100) = P(z > 1.67)$$
$$= P(z \geq 0) - P(0 < z < 1.67)$$
$$= 0.5 - \phi\,(1.67)$$
$$= 0.5 - 0.4525 = 0.0475$$

$\therefore$ **number of bulbs that are likely to last for more than 2100 hours is**

$2500 \times 0.0475 = 118.75 \approx$ **119**

(ii) To find $P(x < 1950)$

If $x = 1950,\ z = -5/6 = -0.83$

$$P(x < 1950) = P(z < -0.83)$$
$$= P(z > 0.83)$$
$$= P(z \geq 0) - P(0 < z < 0.83)$$
$$= 0.5 - \phi(0.83)$$
$$= 0.5 - 0.2967 = 0.2033$$

$\therefore$ **number of bulbs that are likely to last for less than 1950 hours is**

$2500 \times 0.2033 = 508.25 \approx$ **508**

(iii) To find $P(1900 < x < 2100)$

If $x = 1900,\ z = -1.67$ and if $x = 2100,\ z = 1.67$

$$P(1900 < x < 2100) = P(-1.67 < z < 1.67)$$
$$= 2P(0 < z < 1.67)$$
$$= 2\ \phi(1.67) = 2 \times 0.4525 = 0.905$$

$\therefore$ **number of bulbs that are likely to last between 1900 and 2100 hours**

$= 2500 \times 0.905 = 2262.5 \approx$ **2263**

---

67. *In a normal distribution 31% of the items are under 45 and 8% of the items are over 64. Find the mean and S.D of the distribution.*

>> Let μ and σ be the mean and S.D of the normal distribution

By data $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$

We have s.n.v $z = \dfrac{x-\mu}{\sigma}$

When $x = 45,\quad z = \dfrac{45-\mu}{\sigma} = z_1$ *(say)*

$x = 64,\quad z = \dfrac{64-\mu}{\sigma} = z_2$ *(say)*

Thus we have,

$$P(z < z_1) = 0.31 \text{ and } P(z > z_2) = 0.08$$

i.e., $0.5 + \phi(z_1) = 0.31$ and $0.5 - \phi(z_2) = 0.08$

$\Rightarrow$ $\phi(z_1) = -0.19$ and $\phi(z_2) = 0.42$

Refering to the normal probability tables we have

$$0.1915\,(\approx 0.19) = \phi(0.5) \text{ and } 0.4192\,(\approx 0.42) = \phi(1.4)$$

$\therefore$ $\phi(z_1) = -\phi(0.5)$ and $\phi(z_2) = \phi(1.4)$

$\Rightarrow$ $z_1 = -0.5$ and $z_2 = 1.4$

i.e., $\dfrac{45-\mu}{\sigma} = -0.5$ and $\dfrac{64-\mu}{\sigma} = 1.4$

or $\mu - 0.5\sigma = 45$ and $\mu + 1.4\sigma = 64$

By solving we get $\mu = 50,\ \sigma = 10$

Thus **mean** $=$ **50** **and S.D** $=$ **10**

---

68. *In an examination 7% of students score less than 35% marks and 89% of students score less than 60% marks. Find the mean and standard deviation if the marks are normally distributed. It is given that if*

$$P(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2}\,dz \quad \text{then} \quad P(1.2263) = 0.39 \text{ and } P(1.4757) = 0.43$$

>> Let μ and σ be the mean and S.D of the normal distribution.

By data we have $P(x < 35) = 0.07$, $P(x < 60) = 0.89$

We have s.n.v $z = \dfrac{x - \mu}{\sigma}$

When $x = 35$, $z = \dfrac{35 - \mu}{\sigma} = z_1$ (*say*)

$x = 60$, $z = \dfrac{60 - \mu}{\sigma} = z_2$ (*say*)

Hence we have

$$P(z < z_1) = 0.07 \quad \text{and} \quad P(z < z_2) = 0.89$$

i.e., $0.5 + \phi(z_1) = 0.07$ and $0.5 + \phi(z_2) = 0.89$

$\therefore$ $\phi(z_1) = -0.43$ and $\phi(z_2) = 0.39$

Using the given data in the RHS of these we have,

$$\phi(z_1) = -\phi(1.4757) \quad \text{and} \quad \phi(z_2) = \phi(1.2263)$$

$\Rightarrow$ $z_1 = -1.4757$ and $z_2 = 1.2263$

i.e., $\dfrac{35 - \mu}{\sigma} = -1.4757$ and $\dfrac{60 - \mu}{\sigma} = 1.2263$

or $\mu - 1.4757\,\sigma = 35$ and $\mu + 1.2263\,\sigma = 60$

By solving we get $\mu = 48.65$ and $\sigma = 9.25$

Thus **mean = 48.65 and S.D = 9.25**

---

69. *For the following normal distribution find c and also the mean and standard deviation of the frequency distribution.*

$$f(x) = c\,e^{\frac{-1}{24}(x^2-6x+4)}$$

>> We have the p.d.f of the normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2} \qquad \ldots(1)$$

Consider $f(x) = c\,e^{\frac{-1}{24}(x^2-6x+4)}$

i.e., $f(x) = c\,e^{\frac{-1}{24}[(x-3)^2-5]} = c\;e^{5/24}\;e^{-(x-3)^2/24}$

i.e., $f(x) = c\;e^{5/24}\;e^{-(x-3)^2/2(\sqrt{12})^2} \qquad \ldots(2)$

Comparing the exponents of the exponential function in (1) and (2) we have $\mu = 3$ and $\sigma = \sqrt{12}$ provided

$$c\,e^{5/24} = \frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\sqrt{12}\,\sqrt{2\pi}} = \frac{1}{\sqrt{24\pi}}$$

Thus $c = \dfrac{e^{-5/24}}{\sqrt{24\pi}}$; **Mean** $(\mu) = 3$, $S.D(\sigma) = \sqrt{12}$

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70. *Obtain the equation of the normal probability curve that may be fitted to the following data*

Variable	6	7	8	9	10	11	12
Frequency	3	6	9	13	8	5	4

>> The equation of the best fitting normal probability curve is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2}$$

We need to compute μ and σ for the given frequency distribution.

$$\mu = \frac{\sum fx}{\sum f} = \frac{18+42+72+117+80+55+48}{48} = 9$$

$$\sigma^2 = \frac{\sum fx^2}{\sum f} - (\mu)^2$$

$$= \frac{3(36)+6(49)+9(64)+13(81)+8(100)+5(121)+4(144)}{48} - (9)^2$$

$\sigma^2 = 2.5833 \qquad \therefore \qquad \sigma = 1.607$

$$\therefore \quad f(x) = \frac{1}{(1.607)\sqrt{2\pi}} e^{-(x-9)^2/5.1666}$$

Thus the required equation of the normal probability curve is

$$f(x) = (0.24825)\, e^{-(x-9)^2/5.1666}$$

EXERCISES

1. Find the mean and the variance for the following probability distribution choosing k suitably.

x	0	1	2	3	4	5
$p(x)$	k	$5k$	$10k$	$10k$	$5k$	k

2. A random variable X has the following probability mass function.

$X = x_i$	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Determine the value of a and hence find

(i) $P(X < 3)$ (ii) $P(X \geq 3)$ (iii) $P(0 < X < 5)$

Also find the distribution of X

3. Find the value of k such that $p(x) = k/2^x$; $x = 1, 2, 3 \cdots$ represents a probability distribution.
4. Six coins are tossed. Find the probability of getting (i) exactly 3 heads (ii) atleast 3 heads (iii) atleast one head
5. The probability of germination of a seed in a packet of seeds is found to be 0.7. If 10 seeds are taken for experimenting on germination in a laboratory, find the probability that

   (i) 8 seeds germinate (ii) atleast 8 seeds germinate
6. X is a binomially distributed random variable. If the mean and variance of X are 2 and $3/2$ respectively, find the distribution function.
7. 10 coins are tossed 1024 times and the following frequencies are observed. Fit a binomial distribution for the data and calculate the expected frequencies.

No. of heads	0	1	2	3	4	5	6	7	8	9	10
Frequency	2	10	38	106	188	257	226	128	59	7	3

8. A switch board can handle only 4 telephone calls per minute. If the incoming calls per minute follow a Poisson distribution with parameter 3, find the probability that the switch board is over taxed in any one minute.

9. A travel agency has 2 cars which it hires daily. The number of demands for a car on each day is distributed as a Poisson variate with mean 1.5 Find the probability that on a particular day (i) there was no demand (ii) a demand is refused.

10. Fit a Poisson distribution for the following data and calculate the theoretical frequencies.

x	0	1	2	3	4
f	111	63	22	3	1

11. Find the value of k such that the function

$$f(x) = \begin{cases} kx^2, & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function of a continuous random variable. Also find $P(1.5 < x < 2.5)$

12. The p.d.f of a continuous random variable is given by

$$p(x) = \begin{cases} kx(1-x)e^x, & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find k and hence find the mean and the standard deviation.

13. The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is

(i) less than 200 months (ii) between 100 months and 25 years

14. If x is a standard normal variate, find the following probabilities by using normal probability table.

(i) $P(0.87 < x < 1.28)$ (ii) $P(-0.34 < x < 0.62)$

(iii) $P(x > 0.85)$ (iv) $P(x > -0.65)$

15. The mean weight of 500 students during a medical examination was found to be 50 kgs and S.D weight 6 kgs. Assuming that the weights are normally distributed, find the number of students having weight
(i) between 40 and 50 kgs. (ii) more than 60 kgs. given that

$$\phi(1.67) = 0.4525 \text{ where } \phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2}\, dz$$

ANSWERS

1. $k = 1/32$, mean $= 5/2$, variance $= 5/4$
2. $a = 1/81$, $1/9$, $8/9$, $8/27$

0	1	2	3	4	5	6	7	8
1/81	4/81	9/81	16/81	25/81	36/81	49/81	64/81	1

3. $k = 1$
4. 5/16 , 21/32 , 63/64
5. 0.2335 , 0.3828
6. $P(x) = {}^{8}C_{x} (1/4)^{x} (3/4)^{8-x}$
7. ${}^{10}C_{x} (1/2)^{x} (1/2)^{10-x} = \frac{1}{2^{10}} {}^{10}C_{x}$

   Expected frequencies : 1 , 10 , 45 , 120 , 210 , 252 , 210 , 120 , 45 , 10 , 1
8. 0.1847
9. 0.2231 , 0.1912
10. $P(x) = \dfrac{e^{-0.6} (0.6)^{x}}{x!}$; 110 , 66 , 20 , 4 , 1
11. $k = 3/26$; 49/104
12. $k = 1/3 - e$; mean ≈ 0.55, S.D ≈ 0.23
13. 0.6321 , 0.3834
14. 0.0919 , 0.3655 , 0.1977 , 0.7422
15. 226 , 24